Name: Cody Lee

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Math 12 Honours: Section 5.1 Solving Exponential Functions

1. Solve for all values of "x" :

a) $8^{x+1}(32)=128$

$2^{(3x+3)}(2^5)=128$

$2^{3x+8}=2^7$

$x=-\frac{1}{3}$

b) $\left(27^{2x}\right)^3=\frac{1}{729}$

$\left(3^{6x}\right)^3=3^{-6}$

$3^{18x}=3^{-6}$

$x=-\frac{1}{3}$

c) $\frac{8^{-1}+2^{-3}}{64^x}=\left(2^{2x-3}\right)$

$\frac{2^{-3}\cdot 2^{1}}{2^{6x}}=2^{2x-3}$

$2^{-2-6x}=2^{2x-3}$

$2x-\cancel{3}=\overset{1}{\cancel{-2}}-\cancel{6x}$

$-4x$

$x=-\frac{1}{4}$

d) $\frac{256^x}{4^{2x-6}}=\left(2^{2x}\right)^x$

$\frac{2^{8x}}{2^{4x-12}}=2^{2x^2}$

$2^{4x+12}=2^{2x^2}$

$4x+12=2x^2$

$2x^2-\cancel{4}x-\overset{6}{\cancel{12}}=0$

$\frac{2\pm\sqrt{4+24}}{2}$

$=\frac{2\pm2\sqrt{7}}{2}$

$=1\pm\sqrt{7}$

e) $\left(8^{x+4}\right)^x=128^{2x+3}$

$2^{3x^2+12x}=2^{14x+21}$

$3x^2-2x-21=0$

$(3x+7)(x-3)$

$x=-\frac{7}{3}$ or 3

f) $\frac{\left(729^x\right)}{3\times9^{2x-3}}=\left(27^{2x+3}\right)^x$

$\frac{3^{6x}}{3\times3^{4x-6}}=3^{6x^2+9x}$

$3^{\cancel{6x-4x+5}\ 2x+5}$

$6x^2+9x=2x+5$

$6x^2+7x-5=0$

$(3x+5)(2x-1)$

$x=-\frac{5}{3}$ or $\frac{1}{2}$

2. Use Logarithms to solve for "x":

a) $4^{x-3} = 10$ $\log_4 10 = x-3$, $x = 4.66$	b) $6^{2x+1} = 14$ $\log_6 14 = 2x+1$, $x = 0.236$
c) $9^{x^2} = 20$ $\log_9 20 = x^2$, $x = \pm 1.168$	d) $10^{3-x} = 21$ $\log_{10} 21 = 3-x$ $x = 1.678$
e) $(2^{2x})3^x = 8000$ $12^x = 8000$ $\log_{12} 8000 = x$ $x = 3.617$	f) $6^{x^3} = 20$ $\log_6 20$, $x = 1.187$

3. Rewrite each of the following in logarithm form:

$a)\ e^5 = y$ $\log_e y = 5$	$b)\ \cancel{2}a^b = \frac{c}{2}$ $\log_a \frac{c}{2}$ $\log_a c - \log_a 2 = b$	$c)\ 2(3x)^{15} = \frac{y}{2}$ $\log_{3x} \frac{y}{2} = 15$

4. Evaluate each of the following without a calculator:

$a) \log_5 125$ 3	$b) \log_2 128$ 7	$c) \log_9 2183$ 4
$d) \log_{\cancel{16}_{2^4}} 64$ 2^6 $4x=6$ $x=\frac{3}{2}$	$e) \log_7 \sqrt{343}$ $7^{\frac{3}{2}}$ $\frac{3}{2}$	$f) (\log_{\cancel{128}_{2^7}} 1024_{2^{10}})^{-1}$ $7x=10$ $x=\frac{10}{7}$ $\frac{7}{10}$
$g) \log_{8_{2^3}} \left(\frac{1}{4}\right)$ 2^{-2} $3x=-2$ $x=-\frac{2}{3}$	$h) -\log_{2.5} \left(\frac{125}{8}\right)$ -3	$i) \left(\log_{0.125_{2^{-3}}}^{\frac{1}{8}} 64^{\frac{2}{3}}\right)^{-2}$ $-3x=4$ $-\frac{4}{3}$

5. Simplify each of the following logarithms without using a calculator:

a) $\log_6 24 + \log_6 9$ $\log_6 216$ 3	b) $\log_5 100 - \log_5 4$ 2	c) $\log_4 8 + \log_4 64$ $2^3 \times 2^6 = 2^9$ $\frac{9}{2}$

d) $\log_3 324 - \log_3 4$ 3^{81} 4	e) $\log_4 12 + \log_4\left(\frac{2}{3}\right) + 3\log_4 2^3$ $\log_4 64$ 3	f) $\log_8 24 + \cancel{2}\log_8 2 - \log_8 3$ $\frac{248 \times 4}{3} = 32 \quad 2^5$ $2^3 = 8$ $3x = 5 \quad (\log_{2^3} 2^5)$ $x = \frac{5}{3}$
g) $\log_3\sqrt{45} - 0.5\log_3\sqrt{5} + \log_3 9$ $\sqrt{45} \div \sqrt{5} = 3, \quad 3 \times 9 = 27$ 3	h) $\log_3 5! - \log_3 4 - \cancel{1}\log_3 3$ $120 \div 4 = 30$ $\log_3 10$	i) $\log_{x+1}\left(x^3 + 3x^2 + 3x + 1\right)$ 3

6. Solve for all possible values of "x" : : $3^{x-1}\left(9^{\frac{3}{2x^2}}\right) = 27$ $\quad 3^{\frac{6}{2x^2}} \quad 3^3$ by observation we see 1 works

$x^2 - 3x - 3$

$x-1 \,\overline{)\,x^3 - 4x^2 + 3}$

$-(x^3 - x^2)$

$-3x^2 + 3$

$-(-3x^2 + 3x)$

$-3x + 3$

$-(-3x + 3)$

$\frac{3 \pm \sqrt{21}}{2}$

$x - 1 + \frac{6}{2x^2} = 3$

$2x^3 - 2x^2 + 6 = 6x^2$

$2x^3 - 8x^2 + 6 = 0 \quad = x^3 - 4x^2 + 3 = 0$

$x = 1, \frac{3 \pm \sqrt{21}}{2}$

7. Suppose you have the equation $2^{5x-3} = -1$, is it possible to have a solution? Explain and justify your answer:

No, it is impossible to have negative outcome if the base is positive

8. Given the equation, $m^x = m^y$, how many cases are there such that the equation is true?

1. $x = y$
2. $m = 1$
3. $m = -1$, x & y both odd / even

9. Find all values of "x" such that: $6^2\left(6^x\right)^x = \left(6^x\right)\left(6^x\right)\left(6^x\right)$

$6^{x^2+2} = 6^{3x}$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2)$

$x = 1, 2$

10. Solve for all possible values of "x": $4\times9^x-21=25\times3^x$

$4\cdot3^{2x}-21=25\cdot3^x$

$4a^2-25a-21=0$

$(a-7)(4a+3)$ $3^x=7$ or $\cancel{-\frac{3}{4}}$

$x=\log_3 7=1.77$

11. Solve for "x": $2^{x+1}+2^x=2^3+2^5+2^{7-x}$

$2^{x+1}+2^x=\frac{8+32}{40}+\frac{128}{2^x}$

$\cancel{2a+a}\ 3a=40+\frac{128}{a}$

$3a^2-40a-128=0$

$(3a+8)(a-16)$

$2^x=16$

$x=4$

12. Suppose that $P=2^m$ and $Q=3^n$. Which of the following is equal to 12^{mn} for every pair of integers (m,n)?

a) P^2Q b) P^nQ^m c) P^nQ^{2m} d) $P^{2m}Q^n$ e) $P^{2n}Q^m$ (AMC)

$2^{2m}\cdot3^n$ P^2 $2^{mn\cdot2}\cdot3^{mn}$

13. Solve the equation for "y" in terms of "x". For hat values of "x" is there no value of "y" that satisfies the equation: $(x+2)^{2y}\cancel{-x^2}=\cancel{3}\ x^2+3$

$2y=\log_{(x+2)}x^2+3$

$x^2+3>0-3$

$-x+2>0$ $x>-2$

$x\neq-1$

$x^2+3>0-3$

$x\geq\sqrt{3}i$

$x<-2,\ x=-1$

14. Suppose that $60^a=3$ and $60^b=5$. What is the value of $12^{\frac{1-a-b}{2-2b}}$? AMC12

$a=\log_3 60$ $b=\log_5 60$

$a=\frac{\log 60}{\log 3}$ $=\frac{\log 60}{\log 5}$

$60^{a+b}=15$

$1-a-b=4$ $\frac{60}{15}$

$2-2b=\frac{3600}{25}=144$

$\boxed{\log_{12}2}=0.9$

$12=2$

$\frac{\log_{60}4}{\log_{60}144}$

$12^{0.9}=2$

15. Suppose "a", "b", "c" and "d" are positive integers, then what is the smallest possible value of $a+b+c+d$?

$4^a+4^b+4^c+4^d=4^{1023}$

$\cancel{4}\cdot4^x=4^{\cancel{1023}\ 1022}$

1022×4

4088

16. Given the equations below, find all the ordered triples of real values (a,b,c) that satisfy the them:

$c^a = b^{2a}, \quad 2^c = 2(4^a), \quad a+b+c=10$

$2^c = 2^{2a+1}$

$c = 2a+1$

$(2a+1)^a = b^{2a}$

$3a+b=9$

$\cancel{3a+}\sqrt{2a+1} = 9-3a$

$2a+1 = 81+9a^2-54a$

$b^2 = 2a+1$

$b = \sqrt{2a+1}$

$9a^2-56a+80=0$

$(x-4)(9x-20)$

$x=$

$a = 4$ or $\frac{20}{9}$

$\frac{49}{9}$

(a,b,c): $(4, -3, 9)$

$\left(\frac{20}{9}, \frac{7}{3}, \frac{49}{9}\right)$

Name: Cody Lee Date: 11/8/2025

Math 12 Honours: Section 5.2 What are Logs and Basic with Logarithm

$\log(A \times B) = \log A + \log B$ $\log\left(\frac{A}{B}\right) = \log A - \log B$ $\log A^n = n \log A$ $\log_a b^c = \frac{\log b^c}{\log a}$

1. Rewrite each of the following in exponential form:

a) $\log_3 81 = 4$ $3^4 = 81$	b) $\log_4 2048 = b$ $4^b = 2048$	c) $-\log_5 a = 13$ $5^{13} = a^{-1}$
d) $\log_c d = e$ $c^e = d$	e) $\log_{2x} 100 = 5$ $(2x)^5 = 100$	f) $\log_{\sqrt{3}} a = b$ $3^{\frac{b}{2}} = a$

2. Evaluate each of the following without using a calculator:

a) $\log_2 8^3$ 9	b) $\log_3 \sqrt{243}$ $3^{\frac{5}{2}}$ $\frac{5}{2}$	c) $\log 100 + \log_3 \sqrt{3}$ $2 + \frac{1}{2}$ $= 2.5$
d) $3\log_{0.125} 32$ $2^{5 \cdot 3}$ $\frac{1}{8}$ 2^{-3} -5	e) $\log_2 \left(\log(0.0001)\right)^2$ $(-4)^2 = 16$ 4	f) $\log_8\left(\frac{\sqrt{256} \times 64}{\sqrt[3]{1024}}\right)$ $2^{4} \times 2^{6}$ $2^{10} \div 2^{\frac{10}{3}}$ $2^{\frac{20}{3}}$ $(2^3)^{\frac{20}{9}}$ $\frac{20}{9}$
g) $\log_5 25 \times \log_5 0.2 \times \log_{0.2} 125$ 2 -1 $\frac{1}{5}$ (5^{-1}) -3 6	h) $\log_{12} 16 + \log_{12} 9$ 4^2 , 3^2 2	i) $\log_{11}\left(2\log_4 2^{51} + \log 10\right)$ 2^{240} $2^{\cdot 2}$ $+1$ 121 2

3. Expand and rewrite each expression in terms of log A, log B, and or log C

a) $\log\left(\frac{AB}{C}\right)$ $\log A + \log B - \log C$	b) $\log\left(A^3\sqrt{B} \times C^{-1}\right)$ $3\log A + \log\sqrt{B} - \log C$ $3\log A + 0.5\log B - \log C$	c) $\log\left(\frac{10A}{\sqrt{B}}\right) - 0.5\log(100C)$ $\cancel{\log 10} + \log A - (\log\sqrt{B} - \cancel{\log 10} - \log C^{0.5}$ $\log A - \log\sqrt{B} - \log\sqrt{C}$ $\log A - 0.5\log B - 0.5\log C$

d) $\log\left(0.001\sqrt{ab}\right)$ $\log\sqrt{a}+\log\sqrt{b}-3$ $0.5\log a+0.5\log b-3$	e) $\log\left(\dfrac{100a}{\sqrt{c^3}\,b^4}\right)$ $2+\log a-\log\sqrt{c^3}-\log b^4$ $2+\log a-\frac{3}{2}\log c-4\log b$	f) $\log\left(\dfrac{\sqrt{10b}}{a^{3/2}}\right)-\log c^{-1}$ $\frac{1}{2}+0.5\log b-\frac{3}{2}\log a+\log c$
g) $\log\left(\dfrac{a^{-2}}{1000b^2}\right)$ $-2\log a-3-2\log b$	h) $\left[\log_c\left(\dfrac{a^2}{b^3}\right)\right]^{-1}$ $\left(2\log_c a-3\log_c b\right)^{-1}$ $\frac{1}{2}\log_c a-\frac{1}{3}\log_c b$	i) $\dfrac{\log_a b}{\log_c a}\times\dfrac{\log_b ac}{\log_b c}$ $\dfrac{\frac{\cancel{\log b}}{\log a}}{\frac{\log a}{\cancel{\log c}}}\times\dfrac{\frac{\log ac}{\cancel{\log b}}}{\frac{\cancel{\log c}}{\log b}}=\dfrac{\log ac\log b}{2\log a}$

4. Express each of the following as a single logarithm

a) $2\log a+5\log b$ $\log a^2b^5$	b) $3\log x+\dfrac{1}{2}\log y$ $\log x^3\sqrt{y}$	c) $2\log a+\log b-5\log c$ $\log\dfrac{a^2b}{c^5}$
d) $\dfrac{1}{2}\log x-\dfrac{2}{3}\log y$ $\log\dfrac{\sqrt{x}}{\sqrt[3]{y^2}}$	e) $3\log a-4\log b-0.5\log c$ $\log\dfrac{a^3}{b^4\sqrt{c}}$	f) $10\log a-\dfrac{3\log b}{2}$ $\log\dfrac{a^{10}}{\sqrt{b^3}}$
f) $\dfrac{2\log a}{3\log b}+\dfrac{5\log b}{2\log b}$ $\log_b\sqrt[3]{a^2}\sqrt{b^6}$	g) $\dfrac{\log a}{0.4\log c}+\dfrac{\log b}{0.5\log c}$ $\log_c a^{\frac{5}{2}}+\log_c b^2$ $\log_c\sqrt{a^5}\,b^2$	h) $3\log\sqrt{a}+2\log\sqrt[3]{b}-3$ $\log\dfrac{a^{\frac{3}{2}}b^{\frac{2}{3}}}{1000}$ $=\log\dfrac{\sqrt{a^3}\sqrt[3]{b^2}}{1000}$

5. If $\log_3 2 = x$, simplify each logarithm in terms of "x"

$a)\ \log_3 8$

$3x$

$b)\ \log_3 \sqrt{2}$

$\frac{1}{2}x$

$c)\ \log_3 24$

8×3

$3x+1$

$d)\ \log_3 18\sqrt{8}$

$3 \times 3 \times \cancel{2\sqrt{8}} \cdot 2\sqrt{2}$

$4\sqrt{2}$

$2 + 2.5x$

6. Given $\log_3 x = 2$ and $\log_3 y = 5$, evaluate each logarithm

$a)\log_3 xy$

7

$b)\log_3\left(27x^3y\right)$

$\log_3 3^3 + \log_3 x^3 + \log_3 y$

$3 + 6 + 5$

$= 14$

$c)\log_3\left(\frac{3x}{y^3}\right)$

$\log_3 3 + \log_3 x - \log_3 y^3$

$1 + 2 - 15$

-12

$d)\log_3\left(810x^{-2}y\right)$

$3^4 \cdot 10 \cdot x^{-2} \cdot y$

$+4 \qquad 5-4$

$\log_3 10 + 5$

7. Use the fact that $\log_a b = \frac{\log_x b}{\log_x a}$ to simplify the following:

$\frac{1}{\log_y x} \cdot \log_y x$

$a)\ (\log_x y)(\log_y x)$

1

$b)\ (\log_5 8)(\log_8 7)(\log_7 5)$

1

$c)\ \left(\frac{1}{\log_d x}\right)\left(\frac{1}{\log_c x}\right)$

$\log_x d \cdot \log_x c$

$\frac{\log d \log c}{(\log x)^2}$

$d)\ (\log_4 a)(\log_a 2a)(\log_{2a} x)$

$\log_4 x$

$= \frac{1}{2}\log_2 x$

e) $\dfrac{\log_x a}{\log_{xy} a}$

$\dfrac{\cancel{\log a}}{\log x} \times \dfrac{\log xy}{\cancel{\log a}}$ $\dfrac{\log x + \log y}{\log x}$

$1 + \log_x y$

f) $\dfrac{\log_c m}{\log_{cd} m} - \dfrac{\log_c m}{\log_d m}$

$1 + \log_c d - \dfrac{\log m}{\log c} \cdot \dfrac{\log d}{\log m}$

1

8. Solve for "y" $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$

$\log_3 y = \log_x x^2$

$y = 9$

9. If $\log x^5 y^3 = 25$ and $\log \dfrac{x}{y} = 3$, then what is the value of $\log x$?

$5 \log x + 3 \log y = 25$

$\log x - \log y = 3$

$\log x = \log y + 3$

$5(\log y + 3) + 3\log y = 25$

$\cancel{15} + 8(\log y) = \cancel{25}\ 10$

$\cancel{8} \log y = \cancel{10}\ \frac{5}{4}$

$\frac{5}{4} + 3 = \frac{17}{4}$

10. Prove the following identity: $\dfrac{1}{\log_a b} + \dfrac{1}{\log_c b} = \dfrac{1}{\log_{ac} b}$

$\dfrac{\log a}{\log b} + \dfrac{\log c}{\log b} = \dfrac{\log ac}{\log b} = \log_b ac = \dfrac{1}{\log_{ac} b}$

11. Find the value of $\displaystyle\sum_{n=1}^{999} \log \sqrt[3]{\frac{n^2}{n^2+2n+1}}$

$\log \sqrt[3]{\dfrac{n^2}{(n+1)^2} \cdot \dfrac{(n+1)^2}{(n+2)^2} \cdots}$ $= \sqrt[3]{\dfrac{1}{1000^2}}$ $\left(10^{-3}\right)^{\frac{2}{3}} = 10^{-2}$

$= \log 10^{-2} = -2$

12. Simplify the product completely: $\dfrac{\log_2 3}{\log_4 3} \times \dfrac{\log_4 5}{\log_6 5} \times \dfrac{\log_6 7}{\log_8 7} \times \ldots \times \dfrac{\log_{124} 125}{\log_{126} 125} \times \dfrac{\log_{126} 127}{\log_{128} 127}$

$\dfrac{\cancel{\log 3}}{\log 2} \times \dfrac{\cancel{\log 4}}{\cancel{\log 3}} \cdots \times \dfrac{\log 128}{\cancel{\log 127}}$

$= \log_2 128 = 7$

13. If $\log(xy^3)=1$ and $\log(x^2y)=1$. What is $\log(xy)$? Amc 2003 12A

$\log x + 3\log y = 1$, $2\log x + \log y = 1$ $\log x + \log y = ?$

$\log x - 2\log y = 0$ $\log x = 2\log y$ $3\log y = \frac{3}{5}$

$5\log y = \frac{1}{5}$

14. If $a \geq b > 1$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$? AMC 2003 12B

$a = b$

$\log_a a - \log_a b + \log_b b - \log_b a$

$a=b$, $2-1-1 = 0$

15. How many distinct four-tuples (a,b,c,d) of rational numbers are there with:

$a\log_{10}2 + b\log_{10}3 + c\log_{10}5 + d\log_{10}7 = 2005$ Amc 2005 12B

$2^a \cdot 3^b \cdot 5^c \cdot 7^d = 10^{2005}$

$b, d = 0$ 1 pair

$a, c = 2005$

16. The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12th term of the sequence is $\log(b^n)$. what is the value of "n"? amc 12

$2\log a + 5\log b = \cancel{3\log a} + 3\log b$

$2\log b = \log a$

$b^2 = a$, $\log a^3 b^7 = \log b^6 b^7 = \log b^{13}$

$13 + 11(9) = 112$

17. Two distinct numbers "a" and "b" are chosen randomly from the set $\{2, 2^2, 2^3, ...2^{26}\}$. What is the probability that $\log_a b$ is an integer? AMC 12 modified

$$\frac{25+12+7+5+4+3+2+2+1+1+1+1+1}{26 \times 25}$$

$$= \frac{65}{650} = \frac{1}{10}$$

Name: Cody Lee

Date: 8/11/2025

Math 12 Honours: Section 5.3 Solving Logarithmic Equations with Identities

1. Solve for "x" and state all the extraneous roots: Show your work:

a) $\log(6-x)-2\log x=0$

$x>0,\ x<6$

$x^2 = 6-x$

$x^2+x-6=0$

$(x+3)(x-2)$

$x=-3$ or 2

extr. root.

b) $\log_3 x^3-\log_3 3x=3$

$x>0$

$\log_3 x^3-\log_3 3x=\log_3 27$

$\log_3 x^3=\log_3 27+\log_3 3x$

$x^3 = 27\cdot 3x$

$x^{3-2}=81x$

$x=9,\ -9$

root, extr.

c) $\log_2(x-3)+\log_2(x+1)=5$

$x>-1$

$x>3$

$(x-3)(x+1)=32$

$x^2-3x+x-3=32\ -32$

$=x^2-2x-35=0$

$(x-7)(x+5)$

$x=7$ or -5

root extr.

d) $\log_7(x+1)+\log_7 x=\log_7 12$

$x>-1$

$x>0$

$(x+1)(x)=12$

$x^2+x-12=0$

$(x+4)(x-3)=0$

$x=-4$ or 3

extr. root.

e) $\log_4(16x-64)-\log_4(3x-34)=3$

$4^3=64$

$\dfrac{16x-64}{3x-34}=64$

$x>\frac{34}{3}$

$16x-64=192x-2176$

$x>\frac{64}{16}\ 4$

$2112=176x$

$x=12$

f) $\log(x-7)+\log(x+2)=1$

$x>7$

$x>-2$

$(x-7)(x+2)=10$

$x^2-7x+2x-14=10$

$x^2-5x-24=0$

$(x+3)(x-8)=0$

$x=-3$ or 8

extr. root.

g) $\log_3(3x+2)+\log_3 x=\log_3 56$

$x>0$

$(3x+2)(x)=56$

$3x^2+2x-56=0$

$(3x+14)(x-4)$

$x=4$ or $-\frac{14}{3}$

root. extr.

h) $\log x=\frac{2}{3}\log 27+2\log 2-\log 3$

$\dfrac{27^{\frac{2}{3}}\cdot 2^2}{3}=x$

$x=12$

i) $2\log_4 x+\log_4(x-2)-\log_4 2x=1$ $\frac{(x^2)(x-2)}{\cancel{2x}}=4\cdot 8x$ $\quad x>2$, $x>0$ $x^3-2x^2-8x=0$ $\cancel{x}(x^2-2x-8)=0$ $(x-4)(x+2)=0$ $x=4$ or -2 root extr.	j) $\log_2 16^{2x+1}=8$ $2^4 16^{2x+1}=2^3$ $2^{8x+4}=2^3$ $8x+4=3$ -1 $x=-\frac{1}{8}$ //
l) $(\log_8 a)(\log_a 3a)(\log_{3a} x^2)=\log_a a^5$ $\log_8 x^2=5$ $8^5_{2^3}=x^2$ $x=2^{\frac{15}{2}}$ $=128\sqrt{2}$ //	m) $2\log(3-x)=\log 2+\log(22-2x)$ $(3-x)^2=(22-2x)2$ $\quad x<3$, $x<11$ $x^2-6x+9=44-4x$ $x^2-2x-35=0$ $(x-7)(x+5)=0$ $x=7$ or -5 extr. root
n) $2^{\log x^2}=3(2^{1+\log x})+16$ $2^{\log x^2}=3(2\cdot 2^{\log x})+16$ $2^{\log x^2}=6\cdot 2^{\log x}+16$ $A^2=6A+16$ $A^2-6A-16=0$ $(A-8)(A+2)$ $A=8$ or -2 $2^{\log x}=8$, $\log x=3$ $\quad x=1000$ //	o) $\log_5(x+3)+\log_5(x-1)=1$ $(x+3)(x-1)=5$ $x^2+2x-3-5=0$ $\quad x>-3$, $x>1$ $(x+4)(x-2)=0$ $x=-4$ or 2 extr. root
$\log_2(x+2)+5=8+\log_2 x$ $x>-2$, $x>0$ $(x+2)32=256(x)$ $32x+64=\cancel{256}x$ $64=224x$ $x=\frac{\cancel{64}}{\cancel{224}}\ \frac{32}{112}\ \frac{16}{56}\ \frac{8}{28}\ \frac{4}{14}\ \frac{2}{7}$ $=\frac{2}{7}$ //	$2\log x-\log(24-x)=\log 2$ $\frac{x^2}{24-x}=2$ $\quad x>0$, $x<24$ $x^2=48-2x$ $x^2+2x-48=0$ $(x+8)(x-6)=0$ $x=-8$ or 6 extr. root //

$\log_2(2x+4)-\log_2(x-1)=3$	$\log_4 x+\log_2\sqrt{x-2}=1+\log_{16}(x-1)^2$
$\frac{2x+4}{x-1}=8$ $8x-8=2x+4$ $6x$ $x=\frac{5}{6}$	$\sqrt{x}\cdot\sqrt{x-2}=2\sqrt{x-1}$ $\sqrt{x^2-2x}=2\sqrt{x-1}$ $\quad x>0$ $x^2-2x=4x-4$ $x^2-6x+4=0$ $x=\frac{6\pm\sqrt{20}}{2}$ $2\sqrt{5}$ $=3+\sqrt{5}$ or $3-\sqrt{5}$ root. extr.

2. Solve for "x": $\log_a b+\log_{a^2} b=\log_{a^3} b^x$

$b^{\frac{3}{2}}=b^{\frac{x}{3}}$

$\frac{3}{2}=\frac{x}{3}$

$9=2x$

$x=\frac{9}{2}$

3. Solve for "x": $\log_4 x-\log_x 16=\frac{7}{6}-\log_x 8$ [Euclid] $\quad x>0$

$\frac{\log x}{\log 4}-\frac{\log 16}{\log x}=\frac{7}{6}-\frac{\log 8}{\log x}$

$\frac{6\log x}{6\log 4}-\frac{7}{6}=\frac{\log 2}{\log x}$, $\quad \frac{6\log x}{6\log 4}-\frac{7\log x}{6\log 4}=\frac{\log 2}{\log x}$

$\frac{-\log x}{6\log 4}=\frac{\log 2}{\log x}$ $\quad -(\log x)^2=6\log 4\cdot\log 2$

$\log 4^{3/2}\cdot\log 2=\log 2^{13}$

$\cancel{-}(\log x)^2=\log 2^{-13}$

$\log x=\pm\left(\log 2^{\frac{-13}{2}}\right)$

$x=\pm\left(2^{-\frac{13}{2}}\right)$, $2^{\frac{-13}{2}}$ (root), $-\left(2^{\frac{-13}{2}}\right)$ (extr.)

4. Determine all value(s) of "x" that satisfy the equation (find all the extraneous roots) [Euclid]

$\log_{5x+9}(x^2+6x+9)+\log_{x+3}(5x^2+24x+27)=4$

$\log_{5x+9}(x+3)^2+\log_{x+3}(5x+9)(x+3)$

let $\log x+3$ be a

$\log 5x+9$ be b

$\frac{2a}{b}+\frac{b}{a}=3$

$\frac{2a^2+b^2}{ab}=3$

$2a^2-3ab+b^2=0$

$(a-b)(2a-b)=0$

$A=B$:

$x+3=5x+9$

$-6=4x$

$x=-\frac{3}{2}$

$2A=B$

$x^2+6x+9=5x+9$

$x^2+x=0$

$x=\ldots=0$

5. If $\log_2 x$, $(1+\log_4 x)$, $\log_8 4x$ are consecutive terms of a geometric sequence, determine the possible value(s) of "x". Euclid 2009

let $\log_2 x$ be a,

$\log_8 4x = \log_8 4 + \log_8 x$

$\frac{2}{3}$

a, $1+\frac{1}{2}a$, $\frac{2}{3}+\frac{1}{3}a$

$\frac{1+\frac{1}{2}a}{a} = \frac{\frac{2}{3}+\frac{1}{3}a}{1+\frac{1}{2}a}$

$1+\frac{1}{4}a^2+a = \frac{2}{3}a+\frac{1}{3}a^2$

$= \frac{1}{12}a^2 - \frac{1}{3}a - 1 = 0$

$a = 6$ or -2

$\log_2 x = 6$ or -2

$x = 64$ or $\frac{1}{4}$

6. Find the product of the value(s) of "m" that satisfy this equation: $(\log_2 m)^2 + \log_2 m^3 = 10$ Brilliant

$+3\log_2 m$

let $\log_2 m = a$

$a^2+3a-10=0$

$(a+5)(a-2)=0$

$a=-5$ or 2

$m = 2^{-5}$ or 2^2

$2^2 \cdot 2^{-5} = 2^{-3} = \frac{1}{8}$

7. Determine all real numbers "x" for which $\left(\sqrt{x}\right)^{\log_{10} x} = 100$ [Euclid]

$x^{\frac{1}{2}\log_{10} x} = 100$

$\log_x 100 = \frac{1}{\cancel{2}}\log_{10}\sqrt{x}$

$\frac{\log 100}{\log x} = \frac{1\log x}{\cancel{2}\log 10^2}$

$(\log 100) \cdot 2 = (\log x)^2$

$\sqrt{x}$ $\frac{1}{2}\log_{10} x \rightarrow \cancel{2}\frac{1}{2}\log_x 10^2$

x

$100 = x$

$4 = \log x^2$ $\pm 2 = \log x$, $x = 100$ or $\frac{1}{100}$

8. Determine all real numbers "x" for which $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10{,}000$ [Euclid]

let $\log_{10} x = t$

$t^{\log_{10} t} = 10000$

$\log_{10}\left(t^{\log_{10} t}\right) = 4$

$(\log_{10} t)^2 = 4$

$\log_{10} t = \pm 2$, $t = 100$ or $\frac{1}{100}$

$x = 10^{100}$ or $10^{\frac{1}{100}}$

9. Challenge: Given the two equations, find the value of "n" 2003 AIME

$$\log\sin x + \log\cos x = -1 \qquad \text{and} \qquad \log(\sin x+\cos x) = 0.5(\log n - 1)$$

$\log(\sin x \cos x) = -1$

$\log ab = -1$

$\log a + \log b = -1$

$ab = \frac{1}{10} = \log_{10}\frac{1}{10}$

$\log n - \log 10$

$= \left(\log\frac{n}{10}\right)$

$\log 100$

$\log_{10}(a+b) = \left(\frac{1}{2}\log_{10}\frac{n}{10}\right)$

$\sqrt{\frac{n}{10}} = \sin x + \cos x$

$2\left(\frac{1}{10}\right)$

$\frac{n}{10} = \sin^2 x + \cos^2 x + 2\sin x\cos x$

$10\left(1+\frac{2}{10}\right) = 12$

10. Determine the number of ordered pairs (a,b) of integers such that $\log_a b + 6\log_b a = 5$, with $2 \le a \le 2005$, and $2 \le b \le 2005$. 2005 AIME

$\log a = A$

let $\log b = B$

$\log_a b + \log_b a^6 = 5$

$\frac{\log b}{\log a} + \frac{\log a^6}{\log b} = 5$

$\frac{B^2}{AB} + \frac{6A^2}{AB} = 5$

$6A^2 + B^2 = 5AB$

$6A^2 - 5AB + B^2 = 0$

$(3A - B)(2A - B) = 0$

$3A = B$ or $2A = B$

$3\log A^3 = \log B$ $\quad$ $\log A^2 = \log B$

$A^3 = B$ $\quad$ $A^2 = B$

$A: 2 \to 12$ $\quad$ $2 \to 44$

11 $\quad$ 43

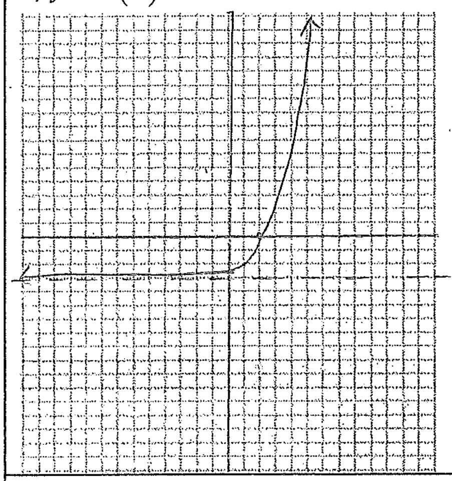
$43 + 11 = 54$

Name: Cody Lee Date: 11/22/2025

Math 12 Honours: Section 5.4 Graphing Exponential & Logarithmic Equations with Transformations

1. For each graph below, find the Y-intercept, X-intercept (if any), Domain and Range, and Asymptotes. Then graph the function with the grid provided. Be sure to label the axis on your grid.

a) $y = 3(2)^{x-2} - 3$



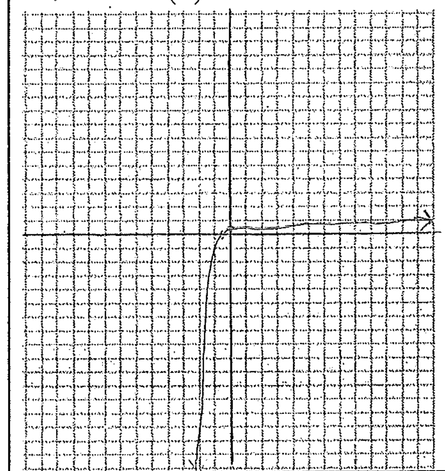
y-int: $(0, -\frac{9}{4})$

x-int: $(2, 0)$

$x \in R$

$y > 3$

b) $y = -0.5(3)^{-x} + 1$



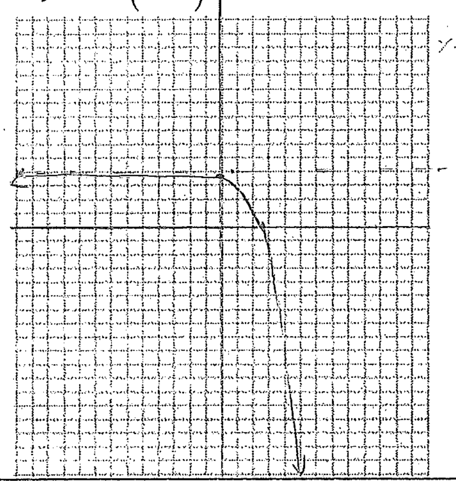
y-int: $(0, 0.5)$

x-int: $(\log_3 \frac{1}{2}, 0)$

$x \in R$

$y < 1$

c) $y = -2(0.\overline{33})^{2-x} + 4$



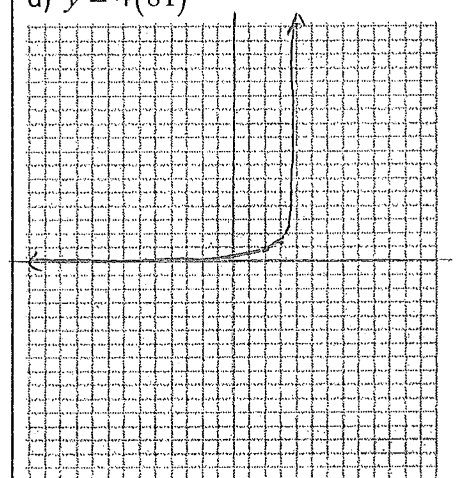
y-int: $(0, \frac{34}{9})$

x-int: $(2 - \log_3 2, 0)$

$x \in R$

$y < 4$

d) $y = 4(81)^{0.25x-1}$



x-int: (N/A)

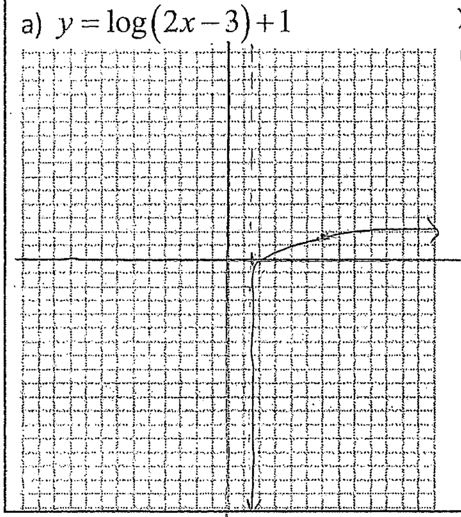
y-int: $(0, \frac{4}{81})$

$x \in R$

$y > 0$

2. Graph each of the following logarithmic functions. Indicate the Domain, Range, equations of Asymptotes, and any intercepts. Be sure to label your axis on the graph:

a) $y = \log(2x-3) + 1$



x-int: $(\frac{31}{20}, 0)$

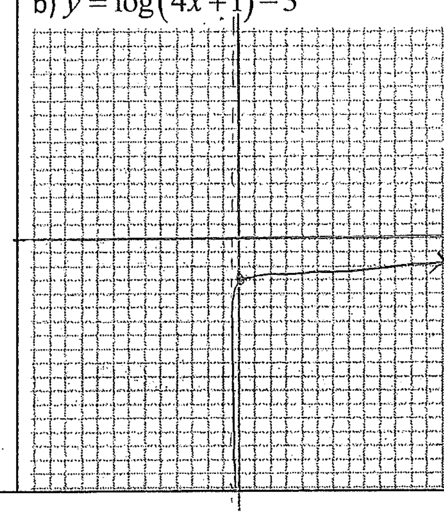
y-int: (N/A)

$y \in R$

$x > \frac{3}{2}$

Asymptote: $x = 1.5$

b) $y = \log(4x+1) - 3$



y-int: $(0, 3)$

x-int: $(249.75, 0)$

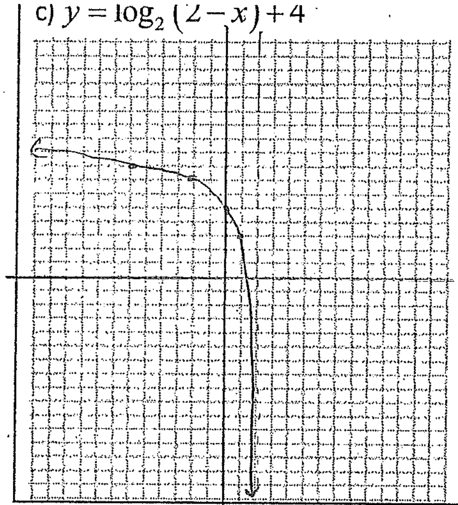
$x > -\frac{1}{4}$

$y \in R$

Asymptote: $x = -\frac{1}{4}$

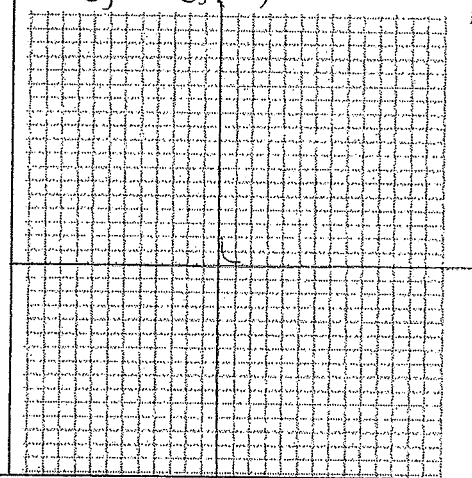
c) $y=\log_2(2-x)+4$

x-int (1.9375, 0)
y-int (0, 5)
$x<2$
$y \in \mathbb{R}$
asymptote: $x=2$



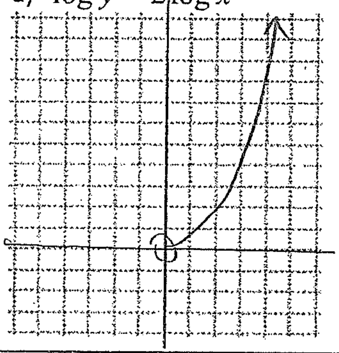
d) $\log y=-\log_3(3x)-5$

no x/y Int-
$x>0$
$y>0$



3. Graph the following functions. Indicate the domain and range:

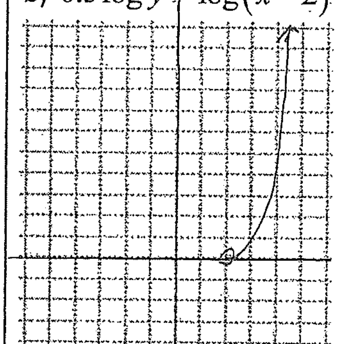
a) $\log y = 2\log x$



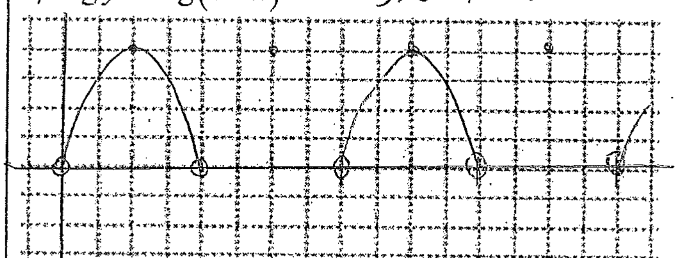
$y>0$
$x>0$

b) $0.5\log y=\log(x-2)$

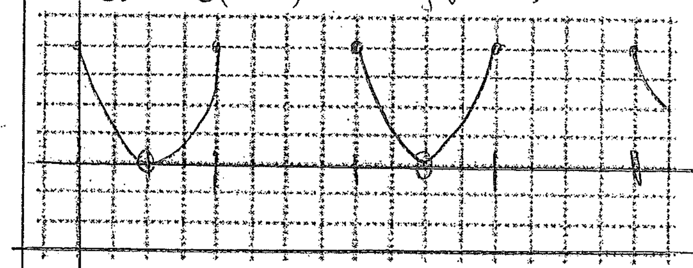
$y=(x-2)^2$
$x>2$
$y>0$



c) $\log y=\log(\sin x)$ $y>0$ $x>0$

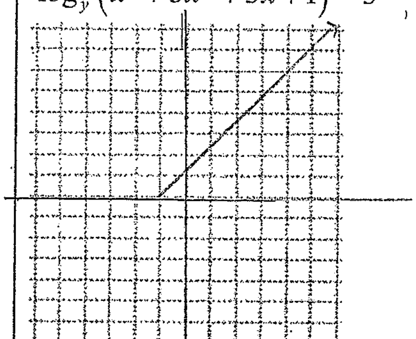


d) $\log y=\log(\sec x)$ $= -\log(\cos x)$



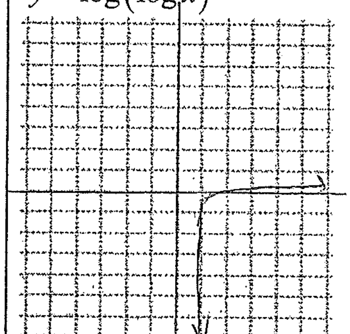
$y=1$
$y=0$

$\log_y(x^3+3x^2+3x+1)=3$, $\log_y(x+1)^3=3$
$y^3=(x+1)^3$



$x>-1$
$y\geq 0$

$y=\log(\log x)$



$x>1$
$y \in \mathbb{R}$

4. Given that $f(x)=\log_3(x+2)-4$, find $f^{-1}(x)$, the inverse function of $f(x)$

$\log_3(y+2)\ \cancel{-4} = x+4$

$3^{x+4}-2=y$ //

5. What transformation is required to go from $y = \log x$ to $y = \log\left(\frac{1}{x}\right)$?

$y \to -y$
(VR)

6. What transformation is required to go from $y = \log_3 x$ to $y = \log_3 \frac{4}{x}$?

1. VR 2. VS $\log_3 4$ up

$y \to -y$; $x \to \frac{1}{x}$
$y \to y - \log_3 4$; $\frac{1}{x} \to \frac{4}{x}$

7. Are the following graphs the same? Yes or NO? Explain:

a) $y = 4(0.5)^x$ and $y = 4(2)^{-x}$

Yes, $0.5 = 2^{-1}$
so $0.5^x = 2^{-x}$

b) $y = 24(0.5)^{2-x}$ and $y = 6(2)^x$

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
$= 6(0.5)^{-x} = 6(2)^x$
same

8. What is the inverse function of $f(x) = 3(5)^{x-2}$? What are the domain, range, x-intercepts, and Y-intercepts of both $f(x)$ and $f^{-1}(x)$? What patterns do you notice?

$x = 3(5)^{y-2}$
$\frac{x}{3} = 5^{y-2}$
$\log_5 \frac{x}{3} = y-2$, $\log_5 x - \log_5 3 + 2 = y$

$f(x)$: $x \in R$, $y > 0$
$f^{-1}(x)$: $x > 0$, $y \in R$
$f(x) \to f^{-1}(x)$ is just switching domain & range

9. Find the coordinates of the points of intersection of the graphs: $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$ (Euclid)

$x = 4$ or $\cancel{-3}$ extr.

$(4, \log 2)$

$\log(x-2) = 1 - \log(x+1)$
$\log(x+1)(x-2) = 1$
$x^2 - x - 2 = 10$

10. Determine all the points where the two functions intersect: $y = \log_{10} x^4$ and $y = (\log_{10} x)^3$ (Euclid)

$\log x = 0$
$x = 1$
$(1, 0)$

$\log x = -2$
$x = \frac{1}{100}$
$(-2, -8)$

$\log x = 2$
$x = 100$
$(2, 8)$

$4a = a^3$ $\log_{10} x = a$

11. If $\log(a+b) = x$ and $\log(a^2 - ab + b^2) = y$, then what is the value of $\sqrt[4]{a^3 + b^3}$ in terms of "x" and "y"?

$x + y = \log_{10}(a^3 + b^3)$

$\sqrt[4]{10^{x+y}}$

12. Solve the inequality

a) $\log_4(2x-3) > 2$

$2x-3 > \cancel{16}\ 19$

$x > 9.5$ or $\frac{19}{2}$

b) $\log_{\frac{1}{4}} x > 5$

$x > \frac{1}{1024}$

13. What is the domain of the following functions:

i) $y = \log_{0.5}\boxed{(\log_5 x)}$ >0

$\log_5 x > 0$

$x > 1$

ii) $y = \log_{0.5}\left(\log_5\left(\log_{\frac{1}{3}} x\right)\right)$

$\log_{\frac{1}{3}} x > 1$

$x > \frac{1}{3}$

14. Solve for "x": $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$ (Euclid)

$\frac{\log x}{2\log 4} - \frac{4\log 2}{\log x} = \frac{7}{6} - \frac{3\log 2}{\log x}$

$\frac{(\log x)^2}{2\log 2} - \frac{\log 2}{\log x} = \frac{7}{6}$

$\frac{(\log x)^2 - 2(\log 2)^2}{2\log x \log 2} = \frac{7}{6}$

$a = 36$

$\frac{36}{16} = \frac{\log x}{\log 2} = 3 \quad \frac{-\frac{2}{3}b}{b} = -\frac{3}{2} \quad \frac{\log x}{\log 2} = -\frac{2}{3}$ $\log 4$ $\quad x^{-3} = 4$

$\boxed{x = 8 \text{ or } \frac{\sqrt[3]{2}}{2}}$

$14\log x \log 2 = 6(\log x)^2 - 12(\log 2)^2$

$\hookrightarrow 36a^2 - 14ab - 12b^2 = 0$

$(3a + 2b)(a - 3b) = 0$

$a = -\frac{2}{3}b \quad a = 3b$

15. Find all the values of "x" such that: $\log_{2x}\left(48\sqrt[3]{3}\right) = \log_{3x}\left(162\sqrt[3]{2}\right)$ (Euclid)

$16 \times 3 \qquad 81 \times 2$

$\frac{\log 2^4 \cdot 3^{\frac{4}{3}}}{\log 2 + \log x} = \frac{\log 2^{\frac{4}{3}} \cdot 3^4}{\log 3 + \log x}$

$\frac{4\log 2 + \frac{4}{3}\log 3}{\log 2 + \log x} = \frac{\frac{4}{3}\log 2 + 4\log 3}{\log 3 + \log x}$

$4\log 2\log 3 + 4\log 2\log x + \frac{4}{3}(\log 3)^2 + \frac{4}{3}\log 3\log x = \frac{4}{3}(\log 2)^2 + 4\log 2\log 3 + 4\log 3\log x + \frac{4}{3}\log 2\log x$

$\frac{4}{3}(\log 3)^2 - \frac{4}{3}(\log 2)^2 = 2\frac{8}{3}\log 3\log x - 2\frac{8}{3}\log 2\log x$

$(\log 3 + \log 2)(\log 3 - \log 2)$

$2\log x(\log 3 - \log 2)$

$\log 6 = 2\log x$

$x = \sqrt{6}$

16. Determine all real numbers "x" for which: $2\log_2(x-1) = 1 - \log_2(x+2)$ (Euclid)

$(x-1)^2(x+2) = 10$

$x^2 - 2x + 1)(x+2) = 10$

$x^3 - 2x^2 - 4x + 2x^2 + x + 2$

$x^3 - 3x + 2 = 10$

$x^3 - 3x - 8 = 0$

$x = \sqrt[3]{4 + \sqrt{15}}$

or

$\sqrt[3]{4 - \sqrt{15}}$

17. Determine all pairs of angles (x,y) with $0° \le x < 180°$ and $0° \le y < 180°$ that satisfy the following systems of equations: (Euclid)

$$\log_2(\sin x \cos y) = -\frac{3}{2} \qquad \text{and} \qquad \log_2\left(\frac{\sin x}{\cos y}\right) = \frac{1}{2}$$

$\sin x \cos y = 2^{-\frac{3}{2}}$

$= \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8} = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$

$\frac{\sin x}{\cos y} = 2^{\frac{1}{2}}$

$= \sqrt{2}$

$ab = \frac{\sqrt{2}}{4}$

$\frac{a}{b} = \sqrt{2}$

$a = b\sqrt{2}$

$b^2\sqrt{2} = \frac{\sqrt{2}}{4}$

$b^2 = \frac{1}{4}, \quad b = \frac{1}{2}$

$\sin x = \frac{\sqrt{2}}{2}, \quad x = 45°$ or $135°$

$\cos y = \frac{1}{2}, \quad y = 60°$

$(45, 60)$ or $(135, 60)$

18. Determine all pairs (a,b) of real numbers that satisfy the following systems of equations: Give your answer in simplified exactd form: (Euclid)

$$\sqrt{a} + \sqrt{b} = 8 \qquad \text{and} \qquad \log_{10} a + \log_{10} b = 2$$

$\log_{10} ab = 2$

$ab = 100 \quad \frac{b}{a}$

$64 = a + b + 2\sqrt{ab}\,100, \quad 2\sqrt{100} = 20$

$a + b = 44 \quad -\frac{b}{a}$

$x^2 - 44x + 100 = 0$

$\frac{44 \pm \sqrt{1536}}{2}$

$\frac{44 \pm 16\sqrt{6}}{2} = 22 \pm 8\sqrt{6}$

(a,b) or (b,a)

19. Solve for "x" $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ (AIME)

$\frac{20}{x} = \frac{2020}{x+3}$

$20x + 60 = 2020x$

$\cancel{(20)}\cancel{60} = \frac{\cancel{2000}}{100}x \quad (\cancel{20})(100)$

$\frac{3}{100} = x$

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Problem

The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution

Let $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020} = n$. Based on the equation, we get $(2^x)^n = 3^{20}$ and $(2^{x+3})^n = 3^{2020}$. Expanding the second equation, we get $8^n \cdot 2^{xn} = 3^{2020}$. Substituting the first equation in, we get $8^n \cdot 3^{20} = 3^{2020}$, so $8^n = 3^{2000}$. Taking the 100th root, we get $8^{\frac{n}{100}} = 3^{20}$. Therefore, $(2^{\frac{3}{100}})^n = 3^{20}$, and using the our first equation($2^{xn} = 3^{20}$), we get $x = \frac{3}{100}$ and the answer is $\boxed{103}$. ~rayfish

Name: Cody Lee Date: 12/6/2025

Math 12 Honours: section 5.5 Applications of Logarithms and Exponential Functions:

1. Suppose you invested \$1000 at 5% compounded annually, how many years will it take your investment to grow to i) \$5000? Ii) \$10,000 iii) \$100,000

$\log_{1.05} 5 = 32.99$ yrs

$\log_{1.05} 10 = 47.19$ yrs

$\log_{1.05} 100 = 94.39$ yrs

2. Calculate the number of years it would take an investment to double for each of the following:

i) 10% compounded semi-annually

$2 = 1\left(1+\frac{0.1}{2}\right)^{2x}$

$\frac{\log_{1.05} 2}{2} = 7.1$ yrs

ii) 10% compounded daily

$2 = 1\left(1+\frac{0.1}{365}\right)^{365x}$

$\frac{\log_{\left(1+\frac{0.1}{365}\right)} 2}{365} = 6.93$ yrs

iii) 8% compounded weekly

$2 = 1\left(1+\frac{0.08}{52}\right)^{52x}$

$\frac{\log_{\left(1+\frac{0.08}{52}\right)} 2}{52} = 8.61$ yrs

iv) 8% compounded monthly

$2 = 1\left(1+\frac{0.08}{12}\right)^{12x}$

$\frac{\log_{\left(1+\frac{0.08}{12}\right)} 2}{12} = 8.69$ yrs

3. In general, for "k%" interest, how many years does an investment take to double? Triple?

$\log_{\left(1+\frac{k}{100}\right)} 2$ (double)

$\log_{\left(1+\frac{k}{100}\right)} 3$ (triple)

4. A 5% investment compounded daily is equivalent to what interest rate compounded annually?

$\left(1+\left(\frac{0.05}{365}\right)\right)^{365} = \left(1+\frac{x}{100}\right)$, $x = 5.13\,\%$

5. Suppose you invest \$5000 each year into an investment that gives a 10% return, how much will your portfolio be after 25 years?

$\frac{5500\left(1.1^{25}-1\right)}{1.1-1} = 540908.83$

6. The half life of sodium-24 is 15.5 hours. Suppose a hospital buys a 100mg sample of sodium-24, how much of the sample will be left after 72 hours?

$$100 \times \frac{1}{2}^{\frac{72}{15.5}} = 4 \text{ mg}$$

7. How long will it take a 100mg sample of sodium-24 to decompose to 1mg?

$$\cancel{100} \times \frac{1}{2}^{\frac{x}{15.5}} = \frac{1}{100}$$

$$15.5\left(\log_{0.5} \frac{1}{100}\right) = 102.98 \text{ hrs}$$

8. Iodine-129 is a dangerous substance in radioactive waste with a half life of 16million years. How long will it take 1,000g of Iondine-129 to decompose to 1g?

$$\cancel{1000} \left(\frac{1}{2}\right)^{\frac{x}{16m}} = \frac{1}{1000} \qquad 16m\left(\log_{0.5} \frac{1}{1000}\right) = 159.45 \text{ million yrs}$$

9. A 500mg radioactive substance decomposed to 20mg in 25 years. What is the half life of the radioactive substance?

$$\cancel{500}\left(\frac{1}{2}\right)^{\frac{25}{x}} = \cancel{20}\ 0.04$$

$$25 \div \left(\log_{0.5} 0.04\right) = 5.38 \text{ yrs}$$

10. A large fossil with 25mg of carbon 14 was found that originally contained 3000mg of carbon 14. If the half life of carbon 14 is 5700 years, then how old is the fossil?

$$\cancel{3000}\left(\frac{1}{2}\right)^{\frac{x}{5700}} = \cancel{25}\ \frac{1}{120} \qquad 5700\left(\log_{0.5} \frac{1}{120}\right) = 39369.28 \text{ yrs}$$

11. 5% of a substance decays in 100 days. What is the half life of the substance?

$$\cancel{1x}\left(\frac{1}{2}\right)^{\frac{100}{x}} =$$

$$100 \div \left(\log_{0.5} 0.95\right) = 1351.34 \text{ days}$$

12. A swarm of locusts can multiply in population by 5 times every 6 weeks. How long will it take a cluster of 5000 locusts to grow to 1 million?

$$\cancel{5000}(5)^{\frac{x}{42}} = \frac{\cancel{1,000,000}}{200} \quad 42(\log_5 200) = 138.27 \text{ days}$$

13. The amount of intensity of an earthquake is measured by the amount of ground motion recorded on a seismograph. An increase in 1unit magnitude on the Ritcher scale represents a 10 fold in intensity. How many times more 'intense' is an earthquake of magnitude 9 than an earthquake of 5.5?

$$10^{9-5.5} = 3162.28 \text{ times}$$

14. For each increase of 1 unit in magnitude, earthquakes are 32 times more powerful in the amount of energy. An earthquake of magnitude of 1 has about $794,328J$ of energy. An earthquake of magnitude 2 has about $25,118,864J$ of energy. An earthquake of 2.5 magnitude has 141,253,754J of energy. How many much energy is in an earthquake with an magnitude of 6.5?

$$141253754 \times 32^4$$
$$= 1.4811 \times 10^{14} J$$

15. In 1976, an earthquake in Italy released approximately 10^{14} joules of energy. What is the magnitude of that earthquake?

$$\log_{32} \frac{10^{14}}{794328} + 1 = 5.38 + 1 = 6.38$$

16. The loudness of sound is measured in decibals (db). Every increase in 10db represents a 10 fold increase in loudness. Going from 10db to 30db, an increase in 20db, is 10^2 times more loud. A regular conversation is about 55db. Listening to a Rock concert is about 110db. How many times louder is the Rock concert than a normal conversation?

$$10^{\frac{(110-55)}{10}} = 316227.8 \text{ times louder}$$

17. A soft whisperer speaks at about 30db. Getting in a yelling match with Cheong is about a thousand times louder. How many db would Cheong be shouting at?

$$30 + 10(\log 1000) = 60 db$$

18. In chemistry, the pH scale measures acidity or alkalinity of a solution. pH ranges from 0 (acidic) to 14 (alkalinity), with 7 being neutral. For each increase in pH, there is a 10 fold increase in alkalinity. For each decrease in pH, there is a 10 fold increase in acidity. Tomato juice has a pH of 4.2 and lemon juice is about 1.8 in pH. How much more acidic is lemon juice than tomato juice?

$10^{(4.2-1.8)} = 251.19$ times more acidic

19. A dishwasher soap has a pH of 12.8. Baking soda is about 9.3 in pH. How much more alkaline is the dishwater soap than baking soda?

$10^{(12.8-9.3)} = 3162.28$ times more alkaline

20. The pH values of milk ranges from 6.4 to 7.6. Pure water has a pH value of 7.0. How much more acidic is milk with a pH of 6.4 than pure water?

$10^{(7.0-6.4)} = 3.98$ times more acidic

21. How much more alkaline is milk with a pH of 7.6 than pure water?

$10^{(7.6-7.0)} = 3.98$ times more alkaline

22. The total amount of arable land in th world is about $3.2\times10^{9}\,ha$. An hectare of land is about 100m by 100m. 0.4 ha of land is enough to grow food for one person. World population in 2021 is 7.874billion, growing at 1.1% annually. When will the demand for arable land exceed the supply available?

$\cancel{0.4} \times 7874\cancel{000},000 \cdot 1.011^{n} = 3.2\cancel{\times10^{9}}$ $1.016...$

$\log_{1.011} 1.016... = n$

$n = 1.45$ years

50%

23. If we are able to reduce the growth rate of the world population by 0.5 and increase the productivity of food production by 200%, then how long will we have until the demand for arable land exceed supply? 300%

land per person $= \frac{0.4}{3}$

growth rate $= 0.55\%$

$\frac{0.4}{3} \times 787\cancel{4000}000 \times 1.0055^{n} = \cancel{3.2}\times10^{9}$

$\log_{1.0055}$

$= 203.19$ yrs

Name: Cody Lee   Date: 13/12/2025

Math 12 Honours: section 5.6 Natural Logarithms and e

1. When should you use the formula $A = P \times e^{rxt}$ versus the final equation: $F = I \times N^{\frac{A}{L}}$
   ↳ continuous growth   ↳ compound interest ↳ population growth ↳ pH levels ↳ decay

2. When a population is decreasing continuously at a rate of 3.5%, then what should the value of "r" be in the equation $A = P \times e^{rt}$?
   r should be -0,035

3. What is the value of the irrational number "e"? How can it be derived?
   $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.71818....$

4. Simplify each of the following into a single exponent:

a) $\frac{2}{e^{-x}}$ — $2e^{x}$	b) $\left(e^{x}\right)^4$ — e^{4x}	c) $e^{1-x}e^{3x}$ — e^{2x+1}
d) $e^{x}e^{-2x}$ — e^{-x}	e) $3e^{2x}\left(1-6e^{-3x}\right)$ — $3e^{2x} - 18e^{-x}$	f) $4e^{3x-2}\left(2-3e^{2x}\right)$ — $8e^{3x-2} - 12e^{5x-2}$

5. Evaluate each of the following without using a calculator

a) $e^{2\ln 4}$ — 16	b) $\ln e^{3}$ — 3	c) $-3\ln e$ — -3
d) $e^{-4\ln 4}$ — $\frac{1}{256}$	e) $\ln\sqrt[4]{e^5}$ — $\frac{5}{4}$	f) $\ln\left(\frac{1}{e}\right)$ — -1
g) $\ln(1+e)$ ← cannot expand — 1.313	h) $e^{\ln 0}$ — undefined	i) $\ln 3 + 2\ln 4 - \ln 48$ — $\ln\frac{3\times 16}{48} = 0$

6. Reduct the following to lowest term

a) $e^{-2\ln 3+3\ln 2}$ — $e^{\ln\frac{8}{9}}$ $\frac{8}{9}$	b) $e^{-\ln\left(\frac{1}{e}\right)}$ — e	c) $\ln\left(3^{-e}e^{3}\right)$ — $-e\ln 3 + 3\ln e$; $3 - e\ln 3$
d) $\ln\left(\frac{\sqrt{\pi}}{e}\right)$ — $\frac{1}{2}\ln\pi - 1$	e) $\ln e^{2\ln e}$ — $(2\ln e)(\ln e)$ 2	f) $e^{\ln 3(\ln 4)}$ — $e^{\ln 4 \ln 3}$

$\frac{1}{2}\ln\pi - 1$

7. Solve for "x"

a) $e^{3x}=4$ $3x=\ln 4$ $x=\frac{\ln 4}{3}$ //	b) $\ln x = 11$ e^{11} //	c) $\ln(3x-2)=4$ $3x-2=e^4$ $\frac{e^4+2}{3}$ //
d) $\ln\left(e^{4-x}\right)=6$ $x=-2$ //	e) $e^{5-3x}=4$ $\frac{\ln 4+5}{3}$ //	f) $\ln x=\ln 11+\ln 6-\ln 3$ 22 //
g) $\ln(\ln x)=4$ $\ln x = e^4$ e^{e^4}	h) $e^{e^{2x}}=4$ $\frac{\ln(\ln 4)}{2}$ //	i) $\frac{\ln\sqrt{x}}{\cancel{2}}=\cancel{3}6\ 12$ e^{12} //
j) $\ln(4x-1)=\frac{\log 10}{\log e}$ $4x-1=\ln 10$ $\frac{11}{4}$ //	k) $\left(e^{\ln 4x}\right)^{\cancel{2}}=4\ 2$ $\frac{1}{2}$ //	l) $\ln x^e=1$ $x^e=e$ $\sqrt[e]{e}$ //
m) $(\ln x)^2-\frac{\log x}{\log e}+6=0$ $\ln x = a$ $a^2-a+6=0$ no real sol. //	n) $e^x-6e^{-x}=1$ $a-\frac{6}{a}=1$ $a^2-a-6=0$ $(a-3)(a+2)$ $e^x=3$ $e^x=-2$ $x=\ln 3$ $x=$ undefined //	o) $(\ln x)^2+(\ln x)=2$ $a^2+a-2=0$ $(a-1)(a+2)$ $x=e$ //
p) $2\ln x=\ln(4x+5)$ $x^2-4x-5=0$ $(x-5)(x+1)$ 5 //	q) $e^x+4e^{-x}=5$ $a+\frac{4}{a}=5a$ $a^2-5a+4=0$ $(a-4)(a-1)$ $\ln 4$ // or $\ln 1$ //	r) $e^{\ln x^3-1}=12$ $x^3-1=12$ $\sqrt[3]{13}$ //

8. Express the following as a single logarithm:

a) $\frac{1}{3}\ln x+2\ln(6x+5)$ $(6x+5)^2(\sqrt[3]{x})$ $(36x^2+60x+30)\sqrt[3]{x}=\ln\left(36\sqrt[3]{x^5}+60\sqrt[3]{x^4}+30\sqrt[3]{x}\right)$ //	b) $3\ln x-\ln\left(\cancel{x^2-1}\right)+\ln\left(\cancel{x^2-1}\right)$ $3\ln x$ //
c) $4\ln x+5\ln(2-x)-2\ln(1+x)$ $\ln\left(\frac{x^4(2-x)^5}{(1+x)^2}\right)$	d) $\frac{2}{3}\ln x-3\ln\left(x^2-2x-8\right)$ $\ln\left(\frac{\sqrt[3]{x^2}}{(x^2-2x-8)^3}\right)$ //

9. The relatiosnip between the elasped time "t", in hours, since Jack took his first does of medication, and the amount of medication M(t), in mg, in his bloodstream is modelled by the following function below.

$M(t) = 30 \times e^{-0.8t}$

I) How much medication will Jack have in his bloodstream after 3 hours?

$30 \times e^{-0.8(3)} = 2.12 \text{ mg}$ //

II) How many hours will it take for Jack to have 1mg left in his bloodstream?

$30 \times e^{-0.8t} = \cancel{1} \ \frac{1}{30}$

$\frac{\ln \frac{1}{30}}{-0.8} = t = 4.25 \text{ hrs}$ //

10. The amount of money Dave has in his investment is given by the formula: $A = Pe^{rt}$. If He invests \$5000 at 2.5% interest, compounded continueously, how long will it take to double his investment?

$\cancel{5000} \times e^{0.025t} = \cancel{10000} \ 2$

$\frac{\ln 2}{0.025} = 27.73 \text{ yrs}$ //

11. A radio-active substance has a half life of 2500 years. What is the equation for the amount of this substance after "t" years in the form of $A = Pe^{rt}$?

equation $= A = P \cdot e^{-0.000277258872t}$ //

$\frac{1}{2} = e^{r(2500)}$ ← $P = 1$, r is YEARLY rate

$\frac{\ln \frac{1}{2}}{2500} = 2500r$, $r = -0.000277258872$

12. TD bank offers an GIC that gives annual interest of 1.5% compounded monthly. What is the equivalent interest rate if the interest is compounded continuously?

$1\left(1 + \frac{0.015}{12}\right)^{12} = 1 \cdot e^{r}$

$e^{r} = \left(1 + \frac{0.015}{12}\right)^{12}$ $r = \ln\left(1 + \frac{0.015}{12}\right)^{12} = 0.01499$

1.499%

13. Each year, Jason's parents contributes \$2500 into his RESP account, then govt will match it with \$500. Suppose the RESP is invested in a fund that gives 8% return annually, compounded continuously, starting when Jason was born, how much will he have in the account when he turns 18?

↑

$\cancel{3000\left(\frac{(1+0.08)^{19} - 1}{(1+0.08) - 1}\right)} = 124338.79$ //

$P \times e^{rt}$

born

18 yrs old

0 1 2 ---> 18

9. The relationsip between the elasped time "t", in hours, since Jack took his first does of medication, and the amount of medication M(t), in mg, in his bloodstream is modelled by

14. When $p=\sum_{k=1}^{6} k\ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p ? AMC 12B

$\ln k^k$

$1^1 \quad 2^2 \quad 3^3 \quad 4^4 \quad 5^5 \quad 6^6$

$2^8 \quad 2^6$

2^{16}

15. Challenge: What is the value of $\lim_{xy\to 1}\left(\frac{\ln x}{\ln y}+\frac{\ln y}{\ln x}\right)=?$

$x, y = 1$

$xy = 1$

$x = \frac{1}{y}$

$$\lim_{xy\to 1}\left(\frac{-\ln \frac{1}{y}y}{\ln y}+\frac{\ln y}{-\ln \frac{1}{y}y}\right)$$

$= -1 + (-1)$

$= -2$

16. Evaluate: $\sum_{n=2}^{\infty}\frac{1}{n(n-1)^3}$, given Euler's beautiful result that: $1+\frac{1}{2^2}+\frac{1}{3^2}+....+\frac{1}{n^2}+....=\frac{\pi^2}{6}$

[Math Circles]

$$\frac{A}{n}+\frac{B}{n-1}+\frac{C}{(n-1)^2}+\frac{D}{(n-1)^3}=\frac{1}{n(n-1)^3}$$

$$A(n-1)^3 + B(n-1)^2 n + C(n-1)n + D(n) = 1$$

$n=0, \ A=-1$

$n=1, \ D=1$

$n=2$

$1 = -1(1)^3 + B(2)(1^2) + C(2)(1) + 1(2)$

$1 = -1 + 2B + 2C + 2 \ 1$

$0 = 2B + 2C$

$B + C = 0$

$n=3$

$8(-1) + B(12) + C(6) + 3 = 1 \ 6$

$2B + C = 1$

$B + C = 0$

$B = 1 \quad C = -1$

$$\underbrace{\frac{-1}{n}+\frac{1}{n-1}}_{1}-\frac{1}{(n-1)^2}+\frac{1}{(n-1)^3}$$

$-\frac{\pi^2}{6}$

Apéry's constant

$\zeta(3)$

$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\dots-\frac{1}{\infty}$

$= 1$

$$\boxed{1-\frac{\pi^2}{6}+\zeta(3)}$$

Name: Cody Lee　　　　Date: 20/12/2025

Math 12 Honours: Section 5.7 Solving Systems with Logarithms

1. Suppose you are given the following solutions to the equation below, which of the solutions are extraneous? Explain:

$$A\log(x+3)+B\log(y+2)=12$$
$$C\log(x-2)-D\log(y-4)=-2$$

solutions: $(4,5)$ ✓✓, $(12,18)$ ✓✓, $(1,4)$ extr, $(4,3)$ extr

$x>-3$, $x>2$ 　 $y>-2$, $y>4$

the domain is $x>2$, and range is $y>4$

2. Solve the following systems of equations:

$$3x+4y+5z=29$$
$$8x-2y+20z=26$$
$$-3x+2y+6z=14$$

$D = \begin{vmatrix} 3 & 4 & 5 \\ 8 & -2 & 20 \\ -3 & 2 & 6 \end{vmatrix}$ $= -30-120-192-36-240+80 = -538$

$D_x = \begin{vmatrix} 29 & 4 & 5 \\ 26 & -2 & 20 \\ 14 & 2 & 6 \end{vmatrix}$ $= 140-1160-624-208+1160+260 = -612$

$D_y = \begin{vmatrix} 3 & 29 & 5 \\ 8 & 26 & 20 \\ -3 & 14 & 6 \end{vmatrix}$ $= -2554$

$D_z = \begin{vmatrix} 3 & 4 & 29 \\ 8 & -2 & 26 \\ -3 & 2 & 14 \end{vmatrix}$ $= -710$

$x = \frac{D_x}{D} = \frac{-612}{-538} = \frac{306}{269}$

$y = \frac{D_y}{D} = \frac{-2554}{-538} = \frac{1277}{269}$

$z = \frac{D_z}{D} = \frac{-710}{-538} = \frac{355}{269}$

3. Given the following systems of equations, for what values of "C" will there be an infinite number of solutions? Explain. For what value(s) of "C" will there be 1 solution OR No solutions. Explain:

$$3x+4y=12$$
$$6x+8y=C$$

if $C=24$, infinite # of solutions because it will be still $3x+4y=12$ & values of x,y will have infinite combination

if $C\neq 24$, there would be only NO solution but not 1 unique solution because $6x+8y=(3x+4y)2$, which in theory would be 24.

4. Given that the systems of equations have one solution, what are the values of "B" and "C":

$$2x+5y=12$$
$$7x+By+C=0$$

$\frac{2}{7} \neq \frac{5}{B}$

$B \neq \frac{35}{2}$, $C \in R$

5. Given each systems of equation, find all sets of real numbers (x,y) that satisfies it:

$\log_2 x+\log_2 y=9$ $\log_4 x+4\log_2 y=22$ $xy=2^9$ $xy^8=2^{44}$ $y^7=2^{35}$ $y=2^5$ 　 $x=2^4$	$5\log_3 x+6\log_3 y=14$ $4\log_3 x-3\log_3 y=6$ $x^5y^6=3^{14}$, $x^5 \cdot 3^4=3^{14}$, $x=3^2$ $\frac{x^4}{y^3}=3^6$ $x=\sqrt[4]{3^6 \times y^3}$ $\sqrt[4]{(3^6 \times y^3)^5} \times y^6=3^{14}$ $3^{7.5}y^6=3^{6.5}$ $y^{9.75}=3^{6.5}$
$\log_x 4+\log_y \sqrt{3}=5$ $\log_x 8-\log_y 9=13$ $2\left(\frac{2\log 2}{\log x}+\frac{\frac{1}{2}\log 3}{\log y}\right)=(5)2$ $\frac{3\log 2}{\log x}-\frac{2\log 3}{\log y}=13$ $\frac{8\log 2}{\log x}+\frac{2\log 3}{\log y}=20$ $\frac{11\log 2}{\log x}=33$ $x^{33}=2^{11}$, $x=\sqrt[3]{2}$ $9+\log_y 9=13-4$ $y^{-4}=3^2$ $y=3^{-\frac{1}{2}}$	$(3x)^{\log 3}=(6y)^{\log 6}$ $6^{\log x}=3^{\log y}$ $\log x \log 6=\log y \log 3$ $\frac{\log x}{\log 3}=\frac{\log y}{\log 6}=k$ $x=3^k$ 　 $y=6^k$ $3^{(k+1)\log 3}=6^{(k+1)\log 6}$ $(\log 3)(k+1)(\log 3)=(\log 6)(\log 6)(k+1)$

$(k+1)[(\log 3)^2-(\log 6)^2]=0$

$k+1=0$, $k=-1$ 　 $x=3^{-1}$, $y=6^{-1}$

$y=3^{\frac{2}{3}}$ 　 $y=9.75$

$\log_8 4x - \log_8 y = \frac{4}{3}$

$4^{\log_4(4x-y)} = \log_2 16$

$\frac{4x}{y} = 16$

$4x - y = 4$

$4x = 16y$

$15y = 4$, $y = \frac{4}{15}$ $x = \frac{16}{15}$

$28y^4 = x^2 + 3$, $28y^4 = y^4 + 3$

$\log_x y^2 = \log_{y^2} x$

$(\log y^2)^2 = (\log x)^2$

$\log y^2 = \log x$

$x = y^2$

$y = 3^{-\frac{1}{2}}$

$x = 3^{-1}$

$\log x - \log 3y = 1$

$3^{3x+y} = 27$

$\frac{x}{3y} = 10$

$x = 30y$

$3^{91y} = 3^3$

$y = \frac{3}{91}$ $x = \frac{90}{91}$

Solve the following system of equation:

$\log x^3 + \log y^2 = 11$ and $\log x^2 - \log y^3 = 3$ Euclid 2003

$2(3a + 2b) = 11(2)$ $2a - 3b = 3$

$6a + 4b = 22$

$6a - 9b = 9$

$13 = 13b$

$b = 1$, $a = 3$

$x = 1000$, $y = 10$

6. Solve the system of equations: $\log x^3 + \log y^2 = 11$, $\log x^2 - \log y^3 = 3$ [Euclid]

$x = 1000$, $y = 10$

7. Determine all the triples (x,y,z) of real numbers that are solutions to the following system of equations [Euclid]:

$\log_9 x + \log_9 y + \log_3 z = 2$ $xyz^2 = 81$

$\log_{16} x + \log_4 y + \log_{16} z = 1$ $xy^2z = 16$

$\log_5 x + \log_{25} y + \log_{25} z = 0$ $x^2yz = 1$

$x^4y^4z^4 = 6^4$

$xyz = 6$

$z = \frac{81}{6}\ \frac{27}{2}$

$y = \frac{16}{6}\ \frac{8}{3}$

$x = \frac{1}{6}$

8. Determine all real solutions to the system of equations and prove that there are no more solutions: Euclid 2008

$$x+\log(x)+1 = y-1$$
$$y+\log(y-1)=z-1$$
$$z+\log(z-2)=x+2$$

$\log(x)(x+\log x)+\log((x+\log x)(\log(x+\log x))-1)=1$

$(x^2+x\log x)+\log(x\log(x+\log x)+\log x\log(x+\log x)-1)=$

ugly, try assume $?=0$, $x^2+x\log x=1$, $x=1$, $y=2$, $z=3$ works!

a

$\cancel{1+x}+\log x+\log(x+\log x)+\log(x+\log x+\log(x+\log x)-1)=\cancel{x+2}$

$\log x+\log a+\log(a+\log(a)-1)=1$

$\log ax+\log(a\log a-1)=1$

9. Determine all real values of "x" for which the equation is true: [Euclid]

$\log_{225}x+\log_{64}y=4$ & $\log_x 225-\log_y 64=1$

$\frac{\log x}{\log 225}+\frac{\log y}{\log 64}=4$

$(\log 64)a+b(\log 225)=4\log 225\log 64$

$\frac{\log 225}{a}-\frac{\log 64}{b}=1$ (A, B)

$\frac{1}{A}-\frac{1}{B}=1$, $B-A=AB$, $A+B=4$

$B=4-A$, $B=1\pm\sqrt5$

$4-A-A=A(4-A)$

$4-2A=4A-A^2$

$A^2-6A+4=0$

$A=\frac{6\pm\sqrt{20}}{2}=3\pm\sqrt5$

solutions: $(225^{3+\sqrt5},64^{1-\sqrt5})$ & $(225^{3-\sqrt5},64^{1+\sqrt5})$

$\frac{\log x}{\log 225}=3\pm\sqrt5$, $225^{3\pm\sqrt5}=x$, $x=225^{3\pm\sqrt5}$, $y=64^{1\pm\sqrt5}$

10. $\sqrt{\log_2 x\cdot\log_2(4x)+1}+\sqrt{\log_2 x\cdot\log_2\left(\frac{x}{64}\right)+9}=4$ Solve the following system of equations: Note There are two solutions: AMC 2002

$\log_2 x\cdot(\log_2 4+\log_2 x)+1$

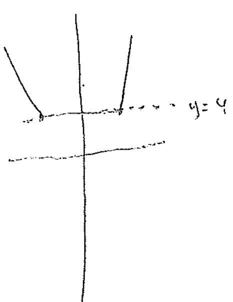
$\log_2 x\cdot(\log_2 x-\log_2 64)+9$

a^2+2a+1

a^2-6a+9

$(a+1)^2$

$(a-3)^2$



$|a+1|+|a-3|=4$

$a\ge-1$, $a\le3$

$\log_2 x\ge-1$, $x\ge\frac12$

$\log_2 x\le3$, $x\le8$

11. Let: $S_1=\{(x,y)\mid\log(1+x^2+y^2)\le1+\log(x+y)\}$, $S_2=\{(x,y)\mid\log(2+x^2+y^2)\le2+\log(x+y)\}$. What is the ratio of the area of S_2 to the area of S_1? 2006 AMC 12A

1. $x^2+y^2+1\le10x+10y$

$x^2-10x+y^2-10y$

$(x-5)^2+(y-5)^2\le49$

2. $x^2+y^2+2\le100x+100y$

$(x-50)^2+(y-50)^2\le4998$

$4998\div49=102$

$102:1$

(x,y,z): $(10000, 1000, 10^{10})$ or $(100, 0.1, 0.01)$

12. Consider the following system of equations, solve the system, (a,b,c) = (-4, 4, -18):

$$(\log x)(\log y) - 3\log 5y - \log 8x = a - 4$$
$$(\log y)(\log z) - 4\log 5y - \log 16z = b\ 4$$
$$(\log z)(\log x) - 4\log 8x - 3\log 625z = c\ -18$$

$ab - 3(\log 5 + b) - (\log 8 + a) = -4$

$ab - 3\log 5 - 3b - \log 8 - a = -4$

$ab - \log 1000 - 3b - a = -4 - 1$

$\left(\frac{3b-1}{b-1}\right)\left(\frac{4b+8}{b-1}\right) - 4\left(\frac{3b-1}{b-1}\right) - 3\left(\frac{4b+8}{b-1}\right) = -6$

$24b + 20(b-1) = 24b^2 + 20b - 24b - 20$

$\frac{12b^2 - 4b + 24b - 8}{(b-1)^2} - \left(\frac{12b-4}{b-1} + \frac{12b+24}{b-1}\right) = -6$

$12b^2 + 20b - 8 - (24b^2 + 4b + 20) = (-6b + 6)(b-1)$

$-12b^2 + 24b + 12 = -6b^2 + 6b + 6b - 6$

$0 = -6b^2 + 12b + 18$

$b = -1$ or 3

$c(b-1) = 4b + 8$

$c = \frac{4b+8}{b-1}$

$b = 3, c = 10, z = 10^{10}$

$b = -1, c = -2, z = 0.01$

$ab - a - 3b = -1$, $a(b-1) - 3b = -1$

$bc - 4b - c = 8$

$ac - 4a - 3c = -6$

$a = \frac{3b-1}{b-1}$

$b = 3, a = 4, x = 10000$

$b = -1, a = 2, x = 100$

$\log y = -1$, $y = 0.1$

$\log y = 3$, $y = 1000$

b) Determine all triples (a,b,c) of real numbers for which the system of equations has an infinite number of solutions (x,y,z)

$\to 0 = 0$

equations:

$ab - a - 3b ~~-3~~ = \boxed{A + 3}$ $\to a(b-1) = \alpha + 3b$

$bc - 4b - c ~~-4~~ = \boxed{B + 4}$ $\to c(b-1) = \beta + 4b$, for it to be $0 = 0$, $b = 1$; $\alpha = -3$, $A = -6$; $\beta = -4$, $B = -8$

$ac - 4a - 3c ~~-12~~ = \boxed{C + 12}$

$c(a-3) - 4a - \sigma = 0$

$a = 3$, $-12 = \sigma$, $C = -24$

$A, B, C = (-6, -8, -24)$

13. Let "S" be the set of ordered triples (x,y,z) of real numbers for which:

$$\log(x+y) = z \quad \text{and} \quad \log(x^2+y^2) = z+1.$$

There are real numbers 'a" and "b" such that for all ordered triples (x,y,z) in "S" have $x^3 + y^3 = a\times 10^{3z} + b\times 10^{2z}$. What is the value of $a+b$? AMC 2005 12B

$k\ ~~10^z~~ = x + y$ $\quad 10 \cdot ~~10^z~~\ k = x^2 + y^2$, $\quad k^2 = x^2 + 2xy + y^2$

$\frac{k^2 - 10k}{2} = xy$

$a(k^3) + b(k^2) = ~~x^3 + y^3~~\ (x+y)(x^2 - xy + y^2)$

$a(k^3) + b(k^2) = k\left(\frac{20k}{2} - \frac{k^2 - 10k}{2}\right)$

$= k\left(\frac{-k^2 + 30k}{2}\right)$

$= \frac{-k^3 + 30k^2}{2} = -\frac{1}{2}k^3 + 15k^2$

$a = -\frac{1}{2}$ $\quad b = 15$

$a + b = 14.5$ or $\frac{29}{2}$

Cody Lee

Logarithms

JV Practice 10/6/19

Elizabeth Chang-Davidson

1 Useful Formulas

- Definition: If $a^x = b$, then $x = \log_a(b)$.
- Convention: log is base 10, ln is base e, though sometimes this is broken.
- Multiplication: $\log_a(xy) = \log_a(x) + \log_a(y)$
- Division:[1] $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
- Exponentiation: $\log_a(x^y) = y\log_a(x)$
- Change of Base: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (key to remembering: the base goes on the bottom)
- Fun identity (follows from Change of Base): $\log_a(b) = \frac{1}{\log_b(a)}$

2 Warmup

$xy^3 = x^2y$, $y^2 = x$, $\log(y^5) = 1$, $\log y^3 = \frac{3}{5}$

1. (2003 AMC 12B Problem 17) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?
2. (2005 AMC 10B Problem 17) Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?

   $(((4^a)^b)^c)^d = 8$, $4^{abcd} = 8$,
3. (2010 AMC 12A Problem 11) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

   $7^{7\log_b 7} \cdot 7^7 = 8^{7\log_b 7}$

   $7^7 = \left(\frac{8}{7}\right)^{7\log_b 7}$

   $b = \frac{8}{7}$

3 Problems

1. (NYCIML F10B25) Compute $(\log_{125} 16)(\log_4 27)(\log_3 625)$.

   $125 = 5^3$, $625 = 5^4$; $\frac{4}{3} \times \frac{3}{2} \times \frac{4}{3} = 8$
2. (NYCIML F06B07) Compute

$$\frac{\log(8)}{\log\frac{1}{8}}$$

   -1

   $\log 70 + \log 20 - 2$
3. (NYCIML S11B26) Let $\log_{10} 70 = m$ and $\log_{10} 20 = p$. Given that $\log_{10} 14 = Am + Bp + C$ where A, B, and C are integers, compute the ordered triple (A, B, C). $(1, 1, -2)$
4. (NYCIML F06A19) If $\log_b(a)\log_c(a)\log_c(b) = 25$ and $\frac{a^2}{c^2} = c^k$, what is the sum of all possible values of k?

[1]Technically this follows from multiplication and exponentiation

$k = 48$

$\log_{c^2} a^2 = 25$

$(c^2)^{25} = a^2$

$c^{50} = a^2$

5. (2018 AMC 12B Problem 7) What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

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6. (2002 AMC 12B Problem 22) For all integers n greater than 1, define $a_n = \frac{1}{\log_n 2002}$. Let $b = a_2 + a_3 + a_4 + a_5$ and $c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}$. What is $b - c$? $\log_{2002} \frac{2\times3\times4\times5}{10\times11\times12\times13\times14} = -1$

$180 + 13$, $143 \times 14 = 1430 + 572 = 2002$

7. (2008 AMC 12A Problem 16) The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

$a^3b^{7} = a^3 b^{3}, b^{13}, b^{22}$ $13 + 99 = 112$

8. (2019 AMC 12A Problem 15) Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

10^{164}

$\sqrt{x} + \sqrt{y} + \frac{1}{2}x + \frac{1}{2}y = 100$, $x + y + \frac{x^2}{2} + \frac{y^2}{2} = 100$

$x^2 + x + y^2 + y = 200$, $(x+1)^2 + (y+1)^2 = 202$, 121, 81, 10 and 8, $x^2 + y^2 = 164$

9. (1984 AIME Problem 5) Determine the value of ab given

$$\log_8 a + \log_4 b^2 = 5$$

$$\log_8 b + \log_4 a^2 = 7$$

$\frac{1}{3}\log_2 a + \log_2 b = 5$, $\log_2 \sqrt[3]{a}\, b = 5$, $\log_2 \sqrt[3]{b}\, a = 7$, $\frac{4}{3}\log_2 a^{\frac{4}{3}} b^{\frac{4}{3}} = 12$, $ab = 2^9 = 512$

10. (David Altizio, Mock AMC 10/12 2013) Suppose x and y are real numbers such that $\log_x(y) = 6$ and $\log_{2x}(2y) = 5$. What is $\log_{4x}(4y)$?

$x^6 = y$, $32x^5 = 2y$, $x = 16$, $\frac{26}{6} = \frac{13}{3}$

11. (2000 AIME II Problem 1) The number $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

$2000 = 125 \times 16 = 5^3 \cdot 2^4$

$\frac{2}{6}\log_{2000} 2^2 + \frac{3}{6}\log_{2000} 5$, $\log_{2000} 2^{\frac{2}{3}} \cdot 5^{\frac{1}{2}} = \frac{1}{6}$, $m + n = 7$

4 Challenge Problems

1. The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

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2. Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x + y) = z \text{ and } \log_{10}(x^2 + y^2) = z + 1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

$a(x+y)^3 + b(x+y)^2 = x^3 + y^3$, $10^{2z} = (x+y)^2$, $10 \cdot 10^z = x^2 + y^2$

3. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

$n = 11$, $792 = 2^3 \times 3^2 \times 11$

4. Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

$8ab - 7a - 6b = 2013$, $(8b - 7)(8a - 6) = 8073$

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5. (David Altizio, Mock AIME I 2015) Suppose that x and y are real numbers such that $\log_x(3y) = \frac{20}{13}$ and $\log_{3x}(y) = \frac{2}{3}$. The value of $\log_{3x}(3y)$ can be expressed in the form $\frac{a}{b}$ where a and b are positive relatively prime integers. Find $a + b$.

$3y = x^{\frac{20}{13}}$, $y = (3x)^{\frac{2}{3}}$

$3 \times (3x)^{\frac{2}{3}} = x^{\frac{20}{13}}$

$39 + 26 = 65$, $65 + 34 = 99$

$\left(3^{\frac{65}{34}}\right)^{\frac{20}{13}}$, $\frac{65}{34} \times \frac{20}{13} = \frac{100}{34}$

$\log_{3^{\frac{99}{34}}} 3^{\frac{100}{34}} = \frac{100}{34} \times \frac{34}{99} = \frac{100}{99}$

$3x = 3^{\frac{99}{34}}$

$a + b = 199$

Cody Lee

$x^2 - 12x + 16 = 0$

$-\frac{b}{a} = 12 \quad \frac{c}{a} = 16 \quad \frac{12 \pm \sqrt{144 - 64}}{2}$

difference $= 4\sqrt{5}$

AIME , AMC 12 Problems on Logarithms 2020 to 2023

Points A and B lie on the graph of $y = \log_2 x$. The midpoint of $\overline{AB}$ is $(6, 2)$. What is the positive difference between the x-coordinates of A and B?

$\textbf{(A)}\ 2\sqrt{11} \qquad \textbf{(B)}\ 4\sqrt{3} \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 4\sqrt{5} \qquad \textbf{(E)}\ 9$

$(1,0)$

x_1	y_1
6	2
x_2	y_2

$\frac{x_1 + x_2}{2} = 6 \quad 12$

$\log_2 x_1 + \log_2 x_2$

A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_2 x$, $\log_3 x$, and $\log_4 x$. What is x?

$\textbf{(A)}\ 2\sqrt{6} \qquad \textbf{(B)}\ 6\sqrt{6} \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 576$

$\log_2 x \cdot \log_3 x \cdot \log_4 x$

$2\log_2 x \log_3 x + 2\log_2 x \log_4 x + 2\log_3 \ldots$

$\frac{1}{\log_x 2} \cdot \frac{1}{\log_x 3} \cdot \frac{1}{\log_x 4}$

$2 \log_x 2 + 2\log_x 3 + 2\log_x 4$

$\log_x 24^2 = 1$

$x = 24^2 = 576$

For how many integers n does the expression

$$\sqrt{\frac{\log(n^2) - (\log n)^2}{\log n - 3}}$$

represent a real number, where log denotes the base 10 logarithm?

$2a > a^2$

$n < 1000$

$\textbf{(A)}\ 900 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 902 \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ 901$

What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where log denotes the base-ten logarithm?

$3\log 2 \qquad -2\log 2$

$(1-\log 2)^3 + (1+\log 2)^3 + (3\log 2)(-2\log 2)$

$= 2(3(\log 2)^2 + 1) - 6(\log 2)^2$

$= 6(\log 2)^2 + 2 - 6(\log 2)^2 = 2$

$\textbf{(A)}\ \frac{3}{2} \qquad \textbf{(B)}\ \frac{7}{4} \qquad \textbf{(C)}\ 2 \qquad \textbf{(D)}\ \frac{9}{4} \qquad \textbf{(E)}\ 3$

What is the value of

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

$\frac{1+3\log 2}{\log 2} \qquad \frac{1+4\log 2}{\log 2}$

$\frac{\log 2}{1+2\log 2} - \frac{\log 2}{1+\log 2}$

$= \frac{2(\log 2)^2}{(\log 2)^2} = 2$

$\textbf{(A)}\ 0 \qquad \textbf{(B)}\ 1 \qquad \textbf{(C)}\ \frac{5}{4} \qquad \textbf{(D)}\ 2 \qquad \textbf{(E)}\ \log_2 5$

What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

$\textbf{(A)}\ 21 \qquad \textbf{(B)}\ 100\log_5 3 \qquad \textbf{(C)}\ 200\log_3 5 \qquad \textbf{(D)}\ 2{,}200 \qquad \textbf{(E)}\ 21{,}000$

What is the product of all the solutions to the equation

$2023 = 7 \times 289$

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023?$$

$\textbf{(A)}\ (\log_{2023} 7 \cdot \log_{2023} 289)^2 \qquad \textbf{(B)}\ \log_{2023} 7 \cdot \log_{2023} 289 \qquad \textbf{(C)}\ 1$

$\textbf{(D)}\ \log_7 2023 \cdot \log_{289} 2023 \qquad \textbf{(E)}\ (\log_7 2023 \cdot \log_{289} 2023)^2$

$\frac{k^2 \log 3}{k \log 5} \cdot \frac{2k \log 5}{2k \log 3}$

$= k$

$= (1+2+3+\ldots 20)(100+\ldots)$

Positive real numbers $b \neq 1$ and n satisfy the equations

$$\sqrt{\log_b n} = \log_b \sqrt{n} \qquad \text{and} \qquad b \cdot \log_b n = \log_b(bn).$$

$n^b = bn$

The value of n is $\frac{j}{k}$, where j and k are relatively prime positive integers. Find $j+k$.

$\left(\frac{5}{4}\right)^4 = \frac{625}{256}$ $625+256 = 881$

$\log_b n = \frac{1}{4}(\log_b n)^2$ $\log_b n = 4$

$a = \frac{1}{4}a^2$ $a = 4$ $b^4 = n$

$(b^4)^b = b(b^4)$

$b^{4b} = b^5$, $4b = 5$, $b = \frac{5}{4}$

There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x). \quad = k$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(\frac{m}{n})$, where m and n are relatively prime positive integers. Find $m+n$.

$(20x)^k = 22x$ $(2x)^k = 202x$ $\frac{(20x)^k}{(2x)^k} = \frac{22\cancel{x}}{202\cancel{x}}$, $10^k = \frac{11}{101}$ $11+101 = 112$

There is a unique positive real number x such that the three numbers $\log_8(2x)$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

$\frac{\log_4 x}{\frac{1}{3}\log_2 2x} = \frac{\log_2 x}{\log_4 x}$ $\frac{1}{4}(\log_2 x)^2 = \frac{1}{3}(\log_2 2 + \log_2 x)(\log_2 x)$

$\frac{1}{4}(\log_2 x)^2 = \frac{1}{3}\lambda + \frac{1}{3}(\log_2 x)^2$

$\log_2 x$

$-\frac{1}{4}a^2 = \frac{1}{3}a^2 + \frac{1}{3}a$

$-\frac{1}{12}a^2 = \frac{1}{3}a$

$a = 0$ or -4

$\log_2 x = -4$

$2^{-4} = \frac{1}{16}$

$m+n = 17$

The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

$\frac{20}{x} = \frac{2020}{x+3}$

$~~6~~ 20x = ~~2020~~$

$60 = ~~2000~~ x$

$x = \frac{60}{2000}$ $\frac{3}{100}$

$m+n = 103$

Determine all real values of x for which

$$\sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = 4$$

$\sqrt{a^2 + 2a + 1} + \sqrt{a^2 - 6a + 9}$

$|(a+1)| + |(a-3)| = 4$

$x \leq 8$ or $x \geq \frac{1}{2}$

$a \leq 3$

$a \geq -1$

(b) Suppose that $f(a) = 2a^2 - 3a + 1$ for all real numbers a and $g(b) = \log_{\frac{1}{2}} b$ for all $b > 0$. Determine all θ with $0 \leq \theta \leq 2\pi$ for which $f(g(\sin\theta)) = 0$.

$(2a-1)(a-1)$

$(-2\log_2 b - 1)(-\log_2 b - 1) = 0$

$-\log_2 b = 1$ or $-2\log_2 b = 1$

$b = \frac{1}{2}$ $b = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$\sin\theta = \frac{1}{2}$ $\sin\theta = \frac{\sqrt{2}}{2}$

$\theta = 30$ or 150 $\theta = 45$ or 135

Cody Lee

Art of Problem Solving logarithm Problems: AIME and AMC 12

$\log_a 27 = \log_3 7$ $\quad \frac{\log 27 \cdot \log_3 \log_7 a}{\log_a 7 \; \log 3}$

1. Suppose that "a", "b", and "c" are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$, find the value of the expression: $a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}$ (2009 AIME II)

$27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25}$

$= 343 + 121 + 5$

$= 469$

2. It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where "a", "b", and "c" are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a+b+c$ (AIMe 2002)

$abc = 6^6$

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$b = \sqrt[3]{6^6} = 36$

$c = 48$

$a = \cancel{35}, \cancel{32}, 27, 20$

$a = 27$

3. The expression below can be written as a fraction in lowest terms, simplify it: $\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$ (2000 AIME)

$2 \log_{2000^6} 4 + 3 \log_{2000^6} 5$

$16 \times 125 = 2000$

$\frac{1}{6}$

4. Positive numbers "x", "y", and "z" satisfies the equations below:

$xyz = 10^{81}$ and $(\log x)(\log yz) + (\log y)(\log z) = 468$, find the value of $\sqrt{(\log x)^2 + (\log y)^2 + (\log z)^2}$ (2010 aime)

$ab + ac + bc = 468$

$abc = 81$

$\sqrt{81^2 - 2(468)}$

$= \sqrt{6561 - 936}$

$= \sqrt{5625}$

$= 75$

$468 \times 2 = 936$

5. Given the system of equations below and that it has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$

$\log(2000xy) - (\log x)(\log y) = 4$

$\log(2yz) - (\log y)(\log z) = 1$

$\log(zx) - (\log z)(\log x) = 0$

$a + c \quad c \quad a$

$\cancel{3}\ 4 \log 2 + a + b = 4 + ab$

$b + c + \log 2 = bc + 1$

$a + b + \log 2 = ab + 1$

$a + c = ac$

$a + b - ab = \log 5$

$b + c - bc = \log 5$

$a + c - ac = 0 \quad 2c - c^2 = 0 \quad c = 2$

$a + b - ab - (b + c - bc)$

$= a + b - ab - b - c + bc$

$= a - c - ab + bc$

$1(a - c) - b(a - c)$

$(1 - b)(a - c) = 0$

$b = 1$ or $a = c$

$a + c = ac$

$2 + b - 2b = \log 5$

$-b = \log 5 - 2$

$-b = 2 - \log 5$

$b = \log 5$

$\log y = \cancel{2} - \log 5$

$\log 100$

$y = 5$ or 20

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6. The sequence a1, a2, a3……. Is geometric with a1=a and common ratio 'r", where "a" and "r" are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \ldots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a,r)? (2006 aime)

$\log_8(a_1 \cdot a_2 \cdot \ldots \cdot a_{12}) = 2006$ $\quad 1+2+\ldots 11 \quad 6\times 11 = 66$

$a^{12} \cdot r^{66} = 2^{6018}$

$a^2 \cdot r^{11} = 2^{1003}$ $\quad (1, 91)$

$2x + 11y = 1003$ $\quad +11 \; -2$ $\quad$ 46 solution

$2 + 1001 = 1003$ $\quad (496, 1)$

$7\times 11\times 13 \quad 11\times 91$

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7. Suppose "A" and "B" are positive real numbers for which $\log_A B = \log_B A$. If neither "A" nor "B" is 1 and $A \neq B$, find the value of AB.

$\log_A B - \frac{1}{\log_A B} = 0$

$(\log_A B - 1)(\log_A B + 1) = 0$

$\log_A B = -1$

$A^{-1} = B \quad AB = 1 \quad (A \cdot A^{-1} = A^0)$

8. Given that $\log_3 2 \approx 0.631$, find the smallest positive integer "a" such that $3^a > 2^{102}$
{Hint: Show that $\log_3 2^{102} = 102 \log_3 2$}

$a \log 3 > 102 \log 2$

$a > \frac{102 \log 2}{\log 3}$

$102 \times 0.631 \approx 64. \ldots$

$a = 65$

9. Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find the value of "n"? (2003 AIME)

$\log(\cos x \sin x) = -1$ $\quad \log(\sin x + \cos x) = \log\left(\frac{n}{10}\right)^{\frac{1}{2}}$

$\frac{1}{10} \times 2 = \frac{2}{10}$ $\quad \cancel{\sin^2 x + \cos^2 x} + 2\cos x \sin x = \frac{n}{10}$

$\quad 10 \quad \frac{2}{10}$

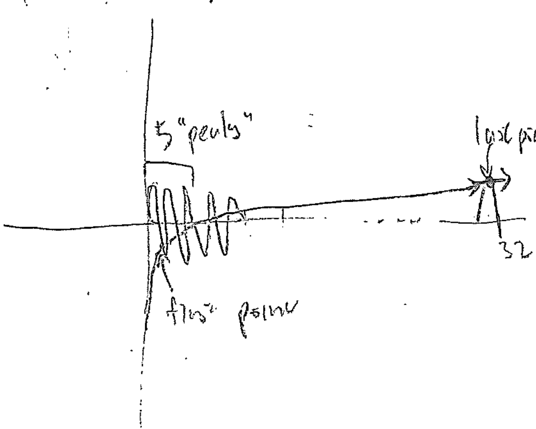
12

10. How many real numbers "x" satisfy the equation: $\frac{1}{5}\log_2 x = \sin(5\pi x)$? (AIME 1991)

(end intersec) $\quad$ (start intersec)

$\frac{1}{5}\log_2 x \leq 1 \quad \frac{1}{5}\log_2 x \geq -1$

$\frac{1}{32} \leq x \leq 32$



5x more frequent/ intersection, but x=0 doesn't count.

$32 \times 5 - 1 = 159$

11. Find the last three digits of the product of the positive roots of the equation below:

$\sqrt{1995}x\log_{1995} x = x^2$ (1995 AIME)

Wrong question,

$\sqrt{1995}\, x^{\log_{1995} x} = x^2$

$1995^{\frac{1}{2}} \cdot x^{\log_{1995} x} = x^2$

$\log_{1995} x = y$

$1995^{y} = x$

$1995^{\frac{1}{2}} \cdot x^{y} = x^2$

$1995^{y^2+\frac{1}{2}} = 1995^{2y}$

$y^2 - 2y + \frac{1}{2} = 0$

$y = \frac{2 \pm \sqrt{2}}{2}$

$= 1 \pm \frac{\sqrt{2}}{2}$

last 3 digits = 025

12. Suppose "x" is in the interval $[0, \frac{\pi}{2}]$ and $\log_{24\sin x}(24\cos x) = \frac{3}{2}$, find the value of $24\cot^2 x$ (AIME 2011)

$0 - 90°$

$(24\sin x)^{\frac{3}{2}} = 24\cos x$

$(24\sin x)^3 = (24\cos x)^2$

$576\sin^3 x = \frac{576}{24}\cos^2 x$

$24^3\sin^3 x = 24^2\cos^2 x$

$24\sin^3 x = 1 - \sin^2 x$

$24a^3 + a^2 - 1 = 0$

test value

$\frac{24}{27} + \frac{1}{9} - 1 = 0$ $\sin x = \frac{1}{3}$

$24 \cdot \frac{1}{\tan^2 x} = 24 \cdot \frac{\cos^2 x}{\sin^2 x}$

$= \frac{576\sin^3 x}{\sin^2 x}$

$= 576\sin x$

192

13. The solutions to the system of equations below are (x_1, y_1) and (x_2, y_2). Find the value of

$\log_{30}(x_1 \times y_1 \times x_2 \times y_2)$ (2002 AIME)

$\log_{225} x + \log_{64} y = 4$

$\log_x 225 - \log_y 64 = 1$

let a be $\log_{15} x$ & b be $\log_8 y$

$a + b = 8$, $\frac{2}{a} - \frac{2}{b} = 1$

$b = 8 - a$, $\frac{2}{a} - \frac{2}{8-a} = 1$

$a^2 - 12a + 16 = 0$

$\log_{30}(15^{12} \cdot 8^{4} \cdot 2^{12})$

12

14. Let "x", "y", and "z" be positive real numbers that satisfy the equation below. If the value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where "p" and "q" are relative prime positive integers, find "p+q".

$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0$ (2012 AIME)

let $x = 2^a$

$y = 2^b$

$z = 2^c$

$\frac{2b+2}{a} = \frac{2c+4}{a+1} = \frac{b+c+3}{4a+1}$

① = ②

$(b+1)(a+1) = (c+2)(a)$

$ab + \cancel{a} + b + 1 = ac + 2\cancel{a}$ ④

① = ③

$(2b+2)(4a+1) = a(b+c+3)$

$8ab + 2b + 8a + 2 = ab + ac + 3a$ ⑤

④ = ⑤

$0 = 6ab + 6a + b + 1$

$(6a+1)(b+1) = 0$

case 1: $6a + 1 = 0$

$a = -\frac{1}{6}$ → plug into ① = ② $(b+1)(\frac{5}{6}) = (c+2)(-\frac{1}{6})$

$\frac{5}{6}b + \frac{5}{6} = -\frac{1}{6}c - \frac{2}{6}$

case 2: $b + 1 = 0$, $b = -1$

case 2: ① = ③

$c + 2 = 0$

$c = -2$

therefore no solution for $b = -1$

we want $a + 5b + c$

using $a = -\frac{1}{6}$

$c = -5b - 7$

$a + 5b + c$

$= a - 7$

$= \frac{-1}{6} - 7 = -\frac{43}{6}$

$p + q = 49$ ← Ans

(Hard) 15. For certain pairs (m,n) of positive integers with $m \geq n$ there are exactly 50 distinct positive integers "k" such that $|\log m - \log k| < \log n$. Find the sum of all possible values of the product mn (2009 AIME)

$-\log n < \log \frac{m}{k} < \log n$

$\frac{1}{n} < \frac{m}{k} < n$

$\frac{m}{n} < k < mn$

$mn-1, mn-2, \dots mn-50$

Then $mn-51 \leq \frac{m}{n} < mn-50$

$m \leq \frac{51n}{n^2-1}$, $m > \frac{50n}{n^2-1}$

n can only be 2 → 7

because of $\frac{51n}{n^2-1} < n$ (8 or above then false)

$n=2$: $\frac{100}{3} < m \leq \frac{102}{3}$, $m=34$

$n=3$ $\frac{150}{8} < m \leq \frac{153}{8}$, $m=19$

$n=4 \to 7$ no integer solution

$34 \cdot 2 + 19 \cdot 3 = 125$

16. The increasing geometric sequence $x_0, x_1, x_2, \dots$ consists entirely of integral powers of 3. Given that $\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57$, find the value of $\log_3(x_{14})$

8 terms   8 terms

$3^{308} = X_0 \cdot X_1 \cdot \dots X_7$

$a^8 r^{28} = 3^{308}$ let $a = 3^n$, $r = 3^k$

① $\boxed{8n + 28k = 308}$

we are asked to find $\log_3(X_{14})$ which is coefficient of ar^{14}

$21 + 5 \times 14 = 91$

$3^{56} \leq ar^7 + ar^6 + \dots + a \leq 3^{57}$

note: notice that if ar^7 is smaller than 3^{56}, the whole sequence sum will be smaller than 3^{56}, same for the max as well

Conclude that $ar^7 = 3^{56}$

plug into ① $4n + 7k = (56)4$

$4n + 28k = 224$

$8n + 28k = 308$

$4n = 84$ $28k = 140$

$n = 21$ $k = 5$

$\boxed{a = 3^{21}}$, $\boxed{r = 3^5}$